GAMMA RINGS WITH DERIVATION

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Abstract
Mainly, in this study the results in the literature obtained on a prime ring are generalized to a prime gamma ring with derivation and the results in the literature obtained on a ring with module valued (σ,τ)-derivation are generalized to on a prime gamma ring with module valued (σ,τ)-derivation respectively. Some certain results getting as the conclusion of this study, have been divided into two parts: First, we assume that M is a prime Γ-ring and d is a non-zero derivation of M. Then, we get some basic specific findings respect to the results which are obtained on a prime ring by Soytürk, M. in 1994. Second, let's assume that the following conditions are provided where X is a nonzero M-bi-module: (G_1) For $x \in X$, $a \in M$ if $x\Gamma M\Gamma a = 0$, then $x = 0$ or $a = 0$, (G_2) For $x \in X$, $a \in M$ if $a\Gamma M\Gamma x = 0$, then $a = 0$ or $x = 0$. In this case, If M is a Γ-ring and the condition (G_1) (or (G_2)) is provided, then we obtain that M must be a prime Γ-ring. Besides, if X is a 2-torsion free then we get that M is also 2-torsion free. Additionally, some results have also been obtained on the commutativity of a prime gamma ring. When M is a non-commutative Γ-ring and $d: M \to X$ is a (σ,τ)-derivation, we get some several findings that they are achieved in parallel with the results obtained previously by Soytürk, M. on a non-commutative ring in 1996. The most important of them is that M must be commutative under the assumption of the condition (G_1).

Keywords: Gamma ring, Prime ring, Derivation, Module valued (σ,τ)-derivation, One-sided ideal.

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1. Introduction

First, E. Posner defined a derivation on an arbitrary ring and gave some properties of a derivation on a Prime Ring [Posner, 1957]. The concept of the gamma ring was first introduced in 1964 by Nobua Nobusawa [Nobusawa, 1964] as follows:
Let M be an additive group with elements a, b, c,… and let Γ be another additive group with elements γ, β, α,... . If the following conditions are provided under the assumptions for $\forall a, b \in M$, $\forall \gamma, \beta \in \Gamma$, $a\gamma b \in M$ and $\gamma a \beta \in \Gamma$ then M is called a Γ-ring.
For $\forall a_1, a_2, a, b_1, b_2, b, c \in M$ ve $\forall \gamma_1, \gamma_2, \gamma, \beta \in \Gamma$

1) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b$,
   $a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$,
   $a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2$
2) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$,
3) For $a, b \in M$ if $a\gamma b = 0$ then $\gamma = 0$.

Later, in 1966, Wilfred E. Barnes removed some of the conditions in the above definition and gave the definition of gamma Ring in the Barnes given in subsection 1.1. Throughout this study, we consider the Γ-ring M in the sense of Barnes. In the literature, studies on gamma rings have been continuing like these [Barnes, 1966], [Jing, 1987], [Kyuno,1978], [Luh, 1969], [Soytürk, 1994], [Soytürk, 1996] [Özkum, 2000], [Öztürk, Jun, 2001] and [Ceran, Aşcı, 2006]… .

The results obtained by M. Soytürk on prime rings with derivation [Soytürk, 1994] and on rings with module valued (σ,τ)-derivation [Soytürk, 1996] generalized to prime gamma rings in Section 2 and gamma rings in Section 3 respectively. We have proved the following results:
Let M be a prime Γ-ring, d be a non-zero derivation of M and $U \neq (0)$ be a right ideal of M with $L = Ann_l U$.

1- The followings are equivalent:
 (i) $d(U) \subseteq L$,
 (ii) $d(U)\Gamma d(U) = 0$,
 (iii) There exist $\exists a \neq 0 \in M$ such that $d(U)\Gamma a = 0$.
 If char $M \neq 2$ then the followings are satisfied:
2- The subring of M generated by d(U) does not contain any nonzero right ideal of M if and only if $d(U) \subseteq L$.
3- Suppose that U is a nonzero ideal of M and g is another non-zero derivation of M such that $gd(U) = 0$ then $d(U) \subseteq L$ and $g(U) \subseteq L$.

Let M be a Γ-ring and U a nonzero ideal of M. Then we shall define the following conditions where X is a nonzero M-bi-module:

(G_1) For $x \in X$, $a \in M$ if $x\Gamma M\Gamma a = 0$, then $x = 0$ or $a = 0$.
(G_2) For $x \in X$, $a \in M$ if $a\Gamma M\Gamma x = 0$, then $a = 0$ or $x = 0$.

4- (i) If (G_1) (or (G_2)) is satisfied, then M is prime.
 (ii) If (G_1) (or (G_2)) is satisfied and X is a 2-torsion free, then M is also 2-torsion free.

5- Let (G_1) (or (G_2)) be satisfied. If $x\Gamma U\Gamma a = (0)$ (or $a\Gamma U\Gamma x = (0)$) for $x \in X$ and $a \in M$, then $x = 0$ or $a = 0$.
6- Let $d : M \rightarrow X$ be a (σ,τ)-derivation and (G_2) be satisfied. Then the followings are verified:
(i) If $d(U) = (0)$, then $d = 0$.
(ii) For $a \in M$ if $a\Gamma d(U) = (0)$, then $a = 0$ or $d = 0$.
7- Let M be non-commutative Γ-ring, $X \neq (0)$ 2-torsion free and (G_1) satisfied. For any (σ,τ)-derivation $d_1 : M \rightarrow X$ and any derivation $d_2 : M \rightarrow M$ where $d_2(U) \subset U$ if $d_1 d_2(U) = (0)$, then $d_1 = 0$ or $d_2 = 0$.
8- Let $X \neq (0)$ and (G_1) be satisfied. For a nonzero (σ,τ)-derivation $d : M \rightarrow X$ if $d(U) \subset C(X)$, then M is commutative.

1. 1. Preliminaries

We give the basic definitions and properties on gamma rings referring [Barnes,1966], [Jing, 1987], [Kyuno,1978], [Luh, 1969] and [Özkum, 2000].

Definition 1.1 [Barnes, 1966] If $M = \{a, b, c,...\}$ and $\Gamma = \{\alpha, \beta,...\}$ are additive abelian groups and for all a, b, c in M and for all α, β in Γ, the followings are satisfied

i) $a\alpha b$ is an element of M,
ii) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$,
iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then M is called a Γ-ring.

Throughout this study we consider the Γ-ring M in the sense of Barnes.

Definition 1.2 Let M be a Γ-ring and $A \subset M$ be a non-zero additive group. If $A\Gamma M \subset A$, then A is called a right ideal of M. Similarly, if $M\Gamma A \subset A$, then A is called a left ideal of M.
If A is both right and left ideal of M then A is called a ideal of M.

A derivation on a Γ-ring were defined by Feng Jie Jing in 1987 as follows:

Definition 1.3 Jing, 1987] Let M be a Γ-ring and d be an additive transformation defined on M. Then, d is called a derivation on the Γ-ring M, if

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$$

for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 1.4 Let M be a Γ-ring. We define the following subset as the center of M

$$Z = \{m \in M \mid c\gamma m = m\gamma c, \forall c \in M \text{ and } \gamma \in \Gamma\}.$$

Lemma 1.1 [Lemma 2; Soytürk, 1994] Let M be a prime Γ-ring. Then the followings are satisfied:
(i) If U is a nonzero right ideal of M and $U \subset Z$ then M is commutative.
(ii) If U is a nonzero right (left) ideal of M and $a \in M$ such that $U\Gamma a = 0$ ($a\Gamma U = 0$) then $a = 0$.
(iii) If U is a nonzero ideal of M and $a, b \in M$ such that $a\Gamma U\Gamma b = 0$ then $a = 0$ or $b = 0$.

Lemma 1.2 [Lemma 3; Soytürk, 1994] Let M be a prime Γ-ring and d be a derivation of M. Then the followings are satisfied:

(i) If U is a nonzero right ideal of M and d(U) = 0 then d = 0.

(ii) If U is a nonzero ideal of M and a ∈ M such that aΓd(U) = 0 (d(U)Γa = 0) then a = 0 or d = 0.

(iii) If Char M ≠ 2 and U ≠ (0) is an ideal of M then $d^2(U) = 0$ implies d = 0.

(iv) If Char M ≠ 2, U ≠ (0) is an ideal of M and d_1, d_2 are two derivations of M such that $d_2(U) \subset U$ and $d_1d_2(U) = 0$ then $d_1 = 0$ or $d_2 = 0$.

Definition 1.5 Let M be a Γ-ring and X an additive abelian group. If the transformation given as X x Γ x M → X, (x, γ, m) → xγm satisfies the following properties for all x, x_1, x_2 ∈ X, γ, δ ∈ Γ and a, b ∈ M

(i) xγ(a + b) = xγa + xγb,

(ii) $(x_1 + x_2)\gamma a = x_1\gamma a + x_2\gamma a$,

(iii) (xγa)δb = xγ(aδb),

then X is defined as a right M-module. Similarly, the left M-module can be defined. If an abelian group X is both a right and a left M-module, then it is called M-bi-module or shortly M-module.

Definition 1.6 Let M be a Γ-ring and σ : M → M be additive mapping. If σ satisfies the equality σ(xαy) = σ(x)ασ(y), for all x, y ∈ M and α ∈ Γ then it is called a Γ-homomorphism of M. Additionally, if σ is one to one and onto then it is called a Γ-automorphism of M.

Definition 1.7 Let M be a Γ-ring and σ, τ : M → M be any two Γ-automorphisms. If the additive mapping d : M → M satisfies the equality d(xαy) = d(x)ασ(y) + τ(x)αd(y), for all x, y ∈ M and α ∈ Γ then it is called a (σ,τ)-derivation of M.

Definition 1.8 Let M be a Γ-ring and X be a M-module then the subset

$$C(X) = \{x \in X \mid x\gamma m = m\gamma x, \forall m \in M \text{ and } \gamma \in \Gamma\}$$

is called the center of X-module (or the center of M on X-module).

And, the subset

$$C_{\sigma,\tau}(X) = \{x \in X \mid x\gamma\sigma(m) = \tau(m)\gamma x, \forall m \in M \text{ and } \gamma \in \Gamma\}$$

is called the (σ,τ)-center of X-module, where σ, τ are any two mappings on M.

Definition 1.9 Let M be a Γ-ring, σ, τ be any two mappings on M and X a nonzero M-bi-module. Then, the additive mapping d : M → X is called as a module valued (σ,τ)-derivation on the Γ-ring M provided that

$$d(m_1\alpha m_2) = d(m_1)\alpha\sigma(m_2) + \tau(m_1)\alpha d(m_2)$$

for all m_1, m_2 ∈ M and α ∈ Γ.

Definition 1.10 [Kyuno, 1978] Let M be a Γ-ring. If the following condition is satisfied for any a, b ∈ M, then M is called a prime gamma ring

$$a\Gamma M\Gamma b = 0 \text{ then } a = 0 \text{ or } b = 0.$$

2. One Sided Ideals and Derivations on Prime Gamma Rings

In this section, we generalize some known results on prime rings with derivation given in [Soytürk, 1994] to prime Γ-rings with derivation.

Let M be a prime gamma ring and U be a nonzero right ideal of M. Then, we denote by L the set of the left-annihilator of U as follows:

$$L = Ann_l\, U = \{x \in M \mid x\Gamma U = 0\}.$$

Remark 2.1: It can be easily seen that L is a left ideal of M.

Lemma 2.1 Let M be a prime gamma ring, d be a nonzero derivation on M, U be a nonzero right ideal of M and $L = Ann_l\, U$. Then the followings are equivalent:

(i) $d(U) \subseteq L$

(ii) $d(U)\Gamma d(U) = 0$,

(iii) There exist $\exists a \neq 0 \in M$ such that $d(U)\Gamma a = 0$.

Proof : **(i) ⇒ (ii):** Assume that $d(U) \subseteq L$. Then $d(U)\Gamma U = 0$. That is,

$$d(u)\alpha v = 0, \text{ for all } u, v \in U \text{ and } \alpha \in \Gamma. \qquad (2.1)$$

Since U is a right ideal then from eq. (2.1), we get for all u, v ∈ U and α, β ∈ Γ

$$\begin{aligned} 0 &= d(u\beta d(w))\,\alpha v \\ &= \{d(u)\beta d(w)) + u\beta d^2(w)\}\,\alpha v \\ &= d(u)\beta d(w)\alpha v + u\beta d^2(w)\,\alpha v \\ &= u\beta d^2(w)\,\alpha v \end{aligned}$$

which means $U\Gamma d^2(U)\Gamma U = 0$. Therefore, from Lemma 1.1 (ii) we have $d^2(U)\Gamma U = 0$.

Using this result and (2.1) we get $0 = d(d(u)\alpha v) = d^2(u)\alpha v + d(u)\alpha d(v) = d(u)\alpha d(v)$, for all u, v ∈ U, α ∈ Γ. Hence we obtain $d(U)\Gamma d(U) = 0$.

(ii) ⇒ (iii): Let $d(U)\Gamma d(U) = 0$. There exist $\exists v \in U$ such that $d(v) \neq 0$. For the proof, we assume the contrary that $d(v) = 0$, for all v ∈ U that is $d(U) = 0$. Then, by Lemma 1.2(i) it implies d = 0, which contradicts our hypothesis. So, there exist $\exists v \in U$ such that $d(v) \neq 0$. And, for this element d(v) from the hypothesis we can write $d(u)\alpha d(v) = 0$, u ∈ U and α ∈ Γ. If we denote d(v) = a then we show that there exist $\exists a \neq 0 \in M$ such that $d(U)\Gamma a = 0$.

(iii) ⇒ (i): We suppose for at least an element a ≠ 0 of M that $d(U)\Gamma a = 0$. Using the definition of U, we can write $0 = d(u\alpha v)\beta a = d(u)\alpha v\beta a + u\alpha d(v)\beta a = d(u)\alpha v\beta a$, for all u, v ∈ U and α, β ∈ Γ. So, $d(U)\Gamma U\Gamma a = 0$. Since M is prime and a ≠ 0 then it implies $d(U)\Gamma U = 0$. Hence, from the definition of L, we get $d(U) \subseteq L$.

Theorem 2.1 Let M be a prime Γ-ring with char M ≠ 2, d be a nonzero derivation on M and U be a nonzero right ideal of M such that $L = Ann_l\, U$. Then, the subring of M generated by d(U) doesn't contain any nonzero right ideal of M if and only if $d(U) \subseteq L$.

Proof: Let S be a subring of M generated by d(U).

⇐: Let $d(U) \subseteq L$. Then

$$d(U)\Gamma U = 0. \qquad (2.2)$$

After from eq. (2.2) and the definition of S, we get $S\Gamma U = 0$. Moreover, S doesn't contain any nonzero right ideal. For the proof, assume that there exist a nonzero right ideal such that $I \subset S$. In this case $I\Gamma U = 0$ and from Lemma 1.1 (ii) we have U = (0) that contradicts our assumption. Hence S does not contain any nonzero right ideal.

⇒: Assume that S doesn't contain any nonzero right ideal of M. Let's define a set such as $T = S \cap U$. At first, $T \neq \emptyset$ due to $0 \in T$. From the hypothesis $d(t)\alpha u = d(t\alpha u) - t\alpha d(u)$, for all $t \in T$, $u \in U$ and $\alpha \in \Gamma$ and then $d(t)\alpha u \in S$ that is $d(T)\Gamma U \subseteq S$. On the other hand, it is easily seen that $d(T)\Gamma U$ is a right ideal of M and it is also included in S. Then it must be zero from the hypothesis, namely $d(T)\Gamma U = 0$. Hence we get

$$d(T) \subseteq L. \qquad (2.3)$$

From the definition of U and S, $u\alpha d(s) \in U$ and $u\alpha d(s) = d(u\alpha s) - d(u)\alpha s \in S$ then $u\alpha d(s) \in S \cap U = T$ for all $u \in U$, $\alpha \in \Gamma$ and $s \in S$. Hence from (2.3) we get $d(u\alpha d(s)) \in L$. And since $d(u\alpha d(s)) = d(u)\alpha d(s) + u\alpha d^2(s)$ then we have for all $u \in U$, $\alpha \in \Gamma$ and $s \in S$

$$d(u)\alpha d(s) + u\alpha d^2(s) \in L. \qquad (2.4)$$

On the other hand, for all $u, v \in U$, $\alpha, \beta \in \Gamma$ and $s \in S$ since

$$\begin{aligned} d(u)\alpha v\beta d(s) &= d(u\alpha v)\, \beta d(s) - u\alpha d(v)\beta d(s) \\ &= d(u\alpha v)\, \beta d(s) - u\alpha d(v)\beta d(s) + (u\alpha v)\, \beta d^2(s) - (u\alpha v)\, \beta d^2(s) \\ &= d(u\alpha v)\, \beta d(s) + (u\alpha v)\, \beta d^2(s) - u\alpha\{d(v)\beta d(s) + v\beta d^2(s)\}. \end{aligned}$$

Eventually, we get

$$d(u)\alpha v\beta d(s) = d(u\alpha v)\, \beta d(s) + (u\alpha v)\, \beta d^2(s) - u\alpha\{d(v)\beta d(s) + v\beta d^2(s)\}. \qquad (2.5)$$

Then, from (2.4) and since L is a left ideal we obtain $d(u)\alpha v\beta d(s) \in L$, for all $u, v \in U$, $\alpha, \beta \in \Gamma$ and $s \in S$. That is, $d(U)\Gamma U\Gamma d(S) \subseteq L$. From the definition of L it implies that $d(U)\Gamma U\Gamma d(S)\Gamma U = 0$ and after using primeness of M then we get

$$d(U)\Gamma U = 0 \ \text{ or } \ d(S)\Gamma U = 0.$$

If $d(U)\Gamma U = 0$, then it is obvious that $d(U) \subseteq L$. Let $d(S)\Gamma U = 0$. In this case since $d(S) \subseteq L$ then $d^2(U) \subseteq L$. Really, in the equality $d^2(u) = d(d(u))$, $u \in U$ using $d(u) \in S$ and $d(S) \subseteq L$ then we get $d^2(u) \in L$, $u \in U$ that implies $d^2(U) \subseteq L$. After, due to $d^2(u\alpha v) = d(d(u\alpha v)) = d(d(u)\alpha v + u\alpha d(v)) = d^2(u)\alpha v + 2\, d(u)\alpha d(v) + u\alpha d^2(v)$, for all $u, v \in U$ and $\alpha \in \Gamma$ then we get $2d(u)\alpha d(v) = d^2(u\alpha v) - d^2(u)\alpha v - u\alpha d^2(v)$. From the assumption $d(S)\Gamma U = 0$ and since $d^2(U) \subseteq L$ and L is a left ideal then we get $2d(u)\alpha d(v) \in L$, for all $u, v \in U$, $\alpha \in \Gamma$. Since char $M \neq 2$ it implies $d(u)\alpha d(v) \in L$, for all $u, v \in U$, $\alpha \in \Gamma$. So

$$d(U)\Gamma d(U) \subseteq L. \qquad (2.6)$$

is obtained. On the other hand, we can write for all $u, v, w \in U$, $\alpha, \beta \in \Gamma$ that $d(u)\alpha w\beta d(v) = d(u\alpha w)\beta d(v) - u\alpha d(w)\beta d(v)$,. Since L is a left ideal and using (2.6) we obtain $d(U)\Gamma U\Gamma d(U) \subseteq L$ and therefore

$$d(U)\Gamma U\Gamma d(U)\Gamma U = 0.$$

Immediately we say from the primeness of M that $d(U)\Gamma U = 0$ and as a result, we get $d(U) \subseteq L$.

Theorem 2.2 Let M be a prime Γ-ring with char M ≠ 2, U be a nonzero ideal of M such that L = Ann_l U and d, g are two nonzero derivations on M. If gd(U) = 0 then d(U) ⊆ L and g(U) ⊆ L.

Proof: Let gd(U) = 0 and S be a subring of M generated by d(U). Then from the definition S, g(S) = 0. Meanwhile, S does not contain any nonzero right ideal of M. Really, we shall assume the opposite that let I ⊆ S be a nonzero ideal of M then g(I) = 0. In this case, from Lemma 1.2 (i) we get g = 0. Whereas it contradicts our hypothesis and so S doesn't contain any nonzero right ideal of M. Then, from the Theorem 2.1 we have

$$d(U) \subseteq L. \tag{2.7}$$

If we use (2.7) in the equality d(uαv) = d(u)αv + uαd(v), for all u, v ∈ U, α ∈ Γ then we get d(uαv) = uαd(v), for all u, v ∈ U, α ∈ Γ that is,

$$d(U\Gamma U) = U\Gamma d(U). \tag{2.8}$$

From the hypothesis and (2.8) we conclude for all u, v ∈ U, α ∈ Γ

$$0 = gd(u\alpha v) = g(u\alpha d(v)) = g(u)\alpha d(v) + u\alpha gd(v)$$
$$= g(u)\alpha d(v),$$

that

$$g(U)\Gamma d(U) = 0. \tag{2.9}$$

Here, d(U) ≠ 0 since d ≠ 0 and U ≠ (0). Really, if d(U) = 0 then Lemma 1.2 (i) implies d = 0 or U = (0) which is a contradiction. Hence, there exist ∃d(U) ≠ 0 ∈ M such that g(U)Γd(U) = 0 then from Lemma 2.1 we have

$$g(U) \subseteq L. \tag{2.10}$$

3. Module Valued (σ,τ)-Derivation on Gamma Rings

In this section, we generalize some known results on rings with module valued (σ,τ)-derivation [Soytürk, 1996] to gamma rings with module valued (σ,τ)-derivation. We mean by the symbol $[\ ,\ ]_\alpha$

$$[a,b]_\alpha = a\alpha b - b\alpha a$$

where a, b ∈ M and α ∈ Γ and we use the following property given in [Lemma 1(i); Soytürk, 1994]

$$[a\beta b,c]_\alpha = a\beta[b,c]_\alpha + [a, c]_\alpha\beta b + a\beta(c\alpha b) - a\alpha(c\beta b)$$

where a, b, c ∈ M and α, β ∈ Γ and d denotes a module valued (σ,τ)-derivation, where σ, τ are any two automorphism satisfying the conditions σαd = dασ, τβd = dβτ, for all α, β ∈ Γ.

Let M be a Γ-ring and X nonzero M-bi-module. We shall define the following conditions:

(G_1) For x ∈ X , a ∈ R if xΓMΓa = 0 then x = 0 or a = 0

(G_2) For x ∈ X , a ∈ R if aΓMΓx = 0 then a = 0 or x = 0

Lemma 3.1 Let M be a Γ-ring and X nonzero M-bi-module then

(i) If (G_1) (or (G_2)) is satisfied, then M is prime.

(ii) If (G_1) (or (G_2)) is satisfied and X is 2-torsion free, then M is also 2-torsion free.

Proof: (i) Let (G_1) be satisfied. Assume that aΓMΓb = 0 for any a, b ∈M. In this case, for given any nonzero x ∈ X we get xΓMΓ(aΓMΓb) = xΓMΓ0 = 0 that is, xΓMΓ(aΓMΓb) = 0 then

$(x\Gamma M\Gamma a)\Gamma M\Gamma b = 0$. It follows from (G_1) that $x\Gamma M\Gamma a = 0$ or $b = 0$ and from (G_1) again $x = 0$ or $a = 0$ or $b = 0$. As $x \neq 0$, we get $a = 0$ or $b = 0$. As a result, when $a\Gamma M\Gamma b = 0$ for any $a, b \in M$ then $a = 0$ or $b = 0$ which is meant that M is prime. By the similar way, it can be shown that when (G_2) exist then M is prime.

(ii) Let (G_1) be satisfied and X be 2-torsion free nonzero M-bi-module. Suppose that $2m = 0$, for any $m \in M$. That is $m + m = 0$. Then for all $x \in X$, $\alpha, \beta \in \Gamma$ and $m' \in M$ we write $x\alpha m'\beta(m + m) = x\alpha m'\beta 0 = 0$ then $x\alpha m'\beta m + x\alpha m'\beta m = 0 \Rightarrow 2x\alpha m'\beta m = 0$, since X is 2-torsion free then we have $x\Gamma M\Gamma m = (0)$, for all $x \in X$ and $m' \in M$. It follows from (G_1) that $x = 0$ or $m = 0$, but $X \neq (0)$ then we get $m = 0$ which yields that M is 2-torsion free. By the similar way, it can be shown that when (G_2) exist then M is 2-torsion free.

Lemma 3.2 Let M be a Γ-ring and X M-bi-module and U a nonzero ideal of M. If (G_1) (or (G_2)) is satisfied and $x\Gamma U\Gamma a = (0)$ (or $a\Gamma U\Gamma x = (0)$) for $x \in X$, $a \in M$ then $x = 0$ or $a = 0$.

Proof: Suppose $x\Gamma U\Gamma a = (0)$, $x \in X$, $a \in M$. Since U is an ideal of M, $x\Gamma M\Gamma U\Gamma a \subset x\Gamma U\Gamma a = (0)$ that is, $x\Gamma M\Gamma U\Gamma a = (0)$ then from (G_1) we get $x = 0$ or $U\Gamma a = (0)$. On the other hand, due to Lemma 3.1 (i) we obtain M as prime. Therefore, when $U\Gamma a = (0)$ by Lemma 1.1 (ii) we get $a = 0$. As a result, we get $x = 0$ or $a = 0$. Similarly, one can get when (G_2) exist and $a\Gamma U\Gamma x = (0)$ that $x = 0$ or $a = 0$.

Lemma 3.3 Let M be a Γ-ring, $U \neq (0)$ a right ideal (or left ideal) of M, $X \neq (0)$ M-bi-module and (G2) (or (G1)) satisfied. Then for a (σ,τ)-derivation $d : M \to X$ the followings are satisfied:

(i) If $d(U) = (0)$ then $d = 0$,

(ii) If for $a \in M$, $a\Gamma d(U) = (0)$ (or $d(U)\Gamma a = 0$) then $a = 0$ or $d = 0$.

Proof: (i) Since U is a right ideal then by the hypothesis $0 = d(u\alpha m) = d(u)\alpha\sigma(m) + \tau(u)\alpha d(m) = \tau(u)\alpha d(m)$, for all $u \in U$, $\alpha \in \Gamma$ and $m \in M$. That is,

$$\tau(u)\alpha d(m) = 0, \qquad (3.1)$$

for all $u \in U$, $\alpha \in \Gamma$ and $m \in M$. If we take m as $\tau^{-1}(s)\beta m$, $s \in M$, $\beta \in \Gamma$ in (3.1) then

$$\begin{aligned} 0 = \tau(u)\alpha d(\tau^{-1}(s)\beta m) &= \tau(u)\alpha\{d(\tau^{-1}(s))\beta\sigma(m) + \tau(\tau^{-1}(s))\beta d(m)\} \\ &= \tau(u)\alpha d(\tau^{-1}(s))\beta\sigma(m) + \tau(u)\alpha s\beta d(m) \\ &= \tau(u)\alpha s\beta d(m) \end{aligned}$$

so,

$$\tau(U)\Gamma M\Gamma d(M) = 0. \qquad (3.2)$$

Applying (G_2) to (3.2) we obtain $\tau(U) = (0)$ or $d(M) = 0$. Then, it follows from the hypothesis that $d = 0$.

(ii) Let $a\Gamma d(U) = (0)$, for $a \in M$. Since U is a right ideal then for all $u \in U$, $\alpha, \beta \in \Gamma$ and $m \in M$

$$\begin{aligned} 0 = a\beta d(u\alpha m) &= a\beta\{d(u)\alpha\sigma(m) + \tau(u)\alpha d(m)\} \\ &= a\beta d(u)\alpha\sigma(m) + a\beta\tau(u)\alpha d(m) \\ &= a\beta\tau(u)\alpha d(m) \end{aligned}$$

that is,

$$a\Gamma\tau(U)\Gamma d(M) = (0) \qquad (3.3)$$

and since $\tau(U)$ is a (right) ideal then by using Lemma 3.2 we conclude it as $a = 0$ or $d = 0$.

Remark 3.1: Similar progress can be doing that when U is a left ideal of M and (G_1) is satisfied then the followings are verified:

(i) If $d(U) = (0)$ then $d = 0$,

(ii) If $d(U)\Gamma a = 0$ for $a \in M$ then $d = 0$ or $a = 0$.

Lemma 3.4 Let M be a non-commutative Γ-ring, $X \neq (0)$ 2-torsion free M-bi-module and $U \neq (0)$ an ideal of M and (G_1) satisfied. If $d_1 : M \to X$ a (σ,τ)-derivation, $d_2 : M \to M$ a derivation such that $d_2(U) \subset U$ and $d_1d_2(U) = (0)$ then $d_1 = 0$ or $d_2 = 0$.

Proof: By hypothesis for $u, v \in U$, $\alpha \in \Gamma$ we get

$$0 = d_1d_2(u\alpha v) = d_1\{d_2(u)\alpha v + u\alpha d_2(v)\} = d_1(d_2(u)\alpha v) + d_1(u\alpha d_2(v))$$
$$= d_1d_2(u)\alpha\sigma(v) + \tau(d_2(u))\alpha d_1(v) + d_1(u)\alpha\sigma(d_2(v)) + \tau(u)\alpha d_1d_2(v)$$
$$= \tau(d_2(u))\alpha d_1(v) + d_1(u)\alpha\sigma(d_2(v)).$$

Here, if we take v as $d_2(v)$ then we get for all $u, v \in U$, $\alpha \in \Gamma$

$$0 = \tau(d_2(u))\alpha d_1(d_2(v)) + d_1(u)\alpha\sigma(d_2(d_2(v))) = \tau(d_2(u))\alpha d_1d_2(v) + d_1(u)\alpha\sigma(d_2^2(v))$$
$$= d_1(u)\alpha\sigma(d_2^2(v)),$$

that is, $d_1(U)\Gamma\sigma(d_2^2(U)) = (0)$. By using Remark 3.1(ii) in this expression we have $d_1 = 0$ or $\sigma(d_2^2(U)) = (0)$ which implies $d_1 = 0$ or $d_2^2(U) = (0)$. If $d_2^2(U) = (0)$ then it follows from Lemma 1.2(iii) that $d_2 = 0$. Eventually, we obtain $d_1 = 0$ or $d_2 = 0$.

Lemma 3.5 Let M be a Γ-ring, $X \neq (0)$ M-bi-module, $U \neq (0)$ an ideal of M and (G_1) exists. If a nonzero (σ,τ)-derivation $d : M \to X$ satisfies $d(U) \subset C(X)$ then M is commutative.

Proof: Since U is an ideal then by the hypothesis $d(u\alpha v) \in C(X)$ for all $u, v \in U$, $\alpha \in \Gamma$. Then it implies $[d(u\alpha v), m]_\beta = 0$ for $\beta \in \Gamma$, for all $u, v \in U$, $\alpha \in \Gamma$, $m \in M$. Thus we can write

$$0 = [d(u)\alpha\sigma(v) + \tau(u)\alpha d(v), m]_\beta$$
$$= [d(u)\alpha\sigma(v),m]_\beta + [\tau(u)\alpha d(v), m]_\beta. \tag{3.4}$$

Arranging the first term of this expression and using the hypothesis we get

$$[d(u)\alpha\sigma(v),m]_\beta = d(u)\alpha[\sigma(v),m]_\beta + [d(u),m]_\beta\,\alpha\sigma(v) + d(u)\alpha(m\beta\sigma(v)) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + d(u)\alpha(m\beta\sigma(v)) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + (m\alpha d(u))\beta\sigma(v) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + m\alpha(d(u)\beta\sigma(v)) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + m\alpha(\sigma(v)\beta d(u)) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + (m\alpha(\sigma(v))\beta d(u) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta + d(u)\beta(m\alpha(\sigma(v)) - d(u)\beta(m\alpha\sigma(v))$$
$$= d(u)\alpha[\sigma(v),m]_\beta$$

eventually,

$$[d(u)\alpha\sigma(v),m]_\beta = d(u)\alpha[\sigma(v),m]_\beta$$

is obtained for all $u, v \in U$, $\alpha \in \Gamma$, $m \in M$. Similarly, if we arrange the second term of eq. (3.4), we obtain the following for all $u, v \in U$, $\alpha \in \Gamma$, $m \in M$

$$[\tau(u)\alpha d(v),m]_\beta = [\tau(u),m]_\beta\alpha d(v).$$

If we write the last expressions in (3.4) then we get for all u, v ∈U, α ∈ Γ, m ∈ M

$$d(u)\alpha[\sigma(v),m]_\beta + [\tau(u),m]_\beta\alpha d(v) = 0. \qquad (3.5)$$

In eq. (3.5) we shall take v as $v\gamma\sigma^{-1}(m)$, γ ∈Γ, then

$0 = d(u)\alpha[\sigma(v\gamma\sigma^{-1}(m)),m]_\beta + [\tau(u),m]_\beta\alpha d(v\gamma\sigma^{-1}(m))$

$= d(u)\alpha[\sigma(v)\gamma m),m]_\beta + [\tau(u),m]_\beta\alpha\{d(v)\gamma m + \tau(v)\gamma d(\sigma^{-1}(m))\}$

$= d(u)\alpha\{ \sigma(v)\gamma[m,m]_\beta + [\sigma(v),m]_\beta\gamma m + \sigma(v)\gamma(m\beta m) - \sigma(v)\beta(m\gamma m) \} + [\tau(u),m]_\beta\alpha d(v)\gamma m + [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m))$

$= d(u)\alpha[\sigma(v),m]_\beta\gamma m + d(u)\alpha\sigma(v)\gamma(m\beta m) - d(u)\alpha\sigma(v)\beta(m\gamma m) + [\tau(u),m]_\beta\alpha d(v)\gamma m + [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m))$

$= \{d(u)\alpha[\sigma(v),m]_\beta+[\tau(u),m]_\beta\alpha d(v)\}\gamma m + [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + d(u)\alpha\sigma(v)\gamma(m\beta m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$,

after using hypothesis and (3.5) we get for all u, v ∈U, α ∈ Γ, m ∈ M

$0 = [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + d(u)\alpha\sigma(v)\gamma(m\beta m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + (\sigma(v)\alpha d(u))\gamma m\beta m - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + \sigma(v)\alpha(m\gamma d(u))\beta m - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + \sigma(v)\alpha m\gamma(m\beta d(u)) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + d(u)\beta(\sigma(v)\alpha m\gamma m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + d(u)\beta(\sigma(v)\alpha m\gamma m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + \sigma(v)\beta d(u)\alpha(m\gamma m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + (\sigma(v)\beta m)\alpha d(u)\gamma m - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$= [\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m)) + d(u)\alpha\sigma(v)\beta(m\gamma m) - d(u)\alpha\sigma(v)\beta(m\gamma m)$

$=[\tau(u),m]_\beta\alpha\tau(v)\gamma d(\sigma^{-1}(m))$.

As a result, for β ∈ Γ we have

$$[\tau(U),M]_\beta\Gamma\tau(U)\Gamma d(\sigma^{-1}(M)) = (0).$$

After, it follows from Lemma 3.2 that $[\tau(U),M]_\beta = 0$ or $d(\sigma^{-1}(M)) = (0)$. Since σ^{-1} is onto then we get $[\tau(U),M]_\beta = 0$ or d(M) = (0). Because of d ≠ 0 then

$$[\tau(U),M]_\beta = 0, \beta \in \Gamma$$

which implies τ(U) ⊂ Z. Then we conclude from the Lemma 1.1(i) that M is commutative, so the proof is complete.

4. Results

In this study, we have obtained some findings on the one sided ideals of Prime Γ-rings with derivative and some findings using Module valued (σ,τ)-derivatives on Γ-rings. In addition to these specific findings on prime gamma rings with derivation, some results on the commutativity of a prime gamma ring are also obtained when M is a non-commutative Γ-ring and d: M → X is a (σ,τ)-derivation that they are achieved in parallel with the results obtained previously by Soytürk, M. on a non-commutative ring in 1996

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