Discussion and application of Monte Carlo method

Xia Liu[1*] and Xiaotong Cai[2]

[1,2]Department of Statistics, School of Mathematics and Statistics, Shandong University of Technology

***Corresponding author**
E-mail addresse:lxia010@163.com

Published: 26 November 2019

Abstract

Monte Carlo method is a statistical method guided by probability and statistics theory. With the progress of computer, Monte Carlo method has been widely applied in multiple integral calculation, nonlinear equations solution, nuclear physics, information science and so on.

In this paper, based on Montmort's problem, we calculate e by Monte Carlo method. Besides, Monte Carlo method is also used to calculate definite integral, now what we are concerned is how to calculate cumulative distribution function $\Phi(x)$ of the standard normal distribution. By generating random numbers for random simulation, we can calculate $\Phi(x)$ and compare it with the original value.

Key words: Monte Carlo method, Stochastic simulation, Standard normal distribution

1. Introduction

Stochastic simulation is also called Monte Carlo method .It is based on the strong law of large numbers[1,2], uses computers to simulate natural phenomenon, through a large number of simulation experiments, and then analysis and inference. The specific test process: first of all, appropriate randomized trials are designed to address the research questions. Then, random sampling is carried out on the system to generate random numbers. Finally, statistics analysis is carried out according to random numbers, and we can obtain some specific parameters of the problems. This is to use stochastic simulation to complete a certain calculation task.

Monte Carlo method is a method of simulating calculation with probability method, which is formed in the famous Buffon's needle problem. It was first posed by George-Louis Leclerc de Buffon, a

Cite this article: Liu, X. & Cai, X. Discussion and application of Monte Carlo method. *European International Journal of Science and Technology*, 8(10), 1-1.

French naturalist. He solved Buffon's needle problem in his book *Essai d'arithmetique morale* published in 1777[3]. The problem is to find the probability that a needle (of length l) thrown at random on a board ruled with a set of equidistant parallel lines b units apart with ($b > l$) will intersect one of the lines. This probability turns out to be $\frac{2l}{\pi b}$ [3,4]. Many mathematicians have calculated the approximate value of π through a large number of injection experiments. With the progress of computer, Monte Carlo method has been widely used in financial engineering, macroeconomics, biomedicine, computational physics (such as particle transport calculation, quantum thermodynamic calculation, nuclear engineering, aerodynamics calculation) and so on. The algebraic reconstruction technique (ART) combined with Monte Carlo method has been proposed for image reconstruction in gamma ray transmission tomography, which is generally used for the nondestructive assay of special nuclear materials [5]. The advantage of this method is that it is easy to be extended from two-dimensional to three-dimensional. Compared with other calculation methods, it is simpler and faster. A.K.Murtazaev and Zh.G.Ibaev have investigated a two-dimensional anisotropic lsing model with competing interactions on a square lattice by combining Monte Carlo method and Wang–Landau algorithm[6].The curves of the density-of-states distribution and the order parameter are obtained[6]. In the field of mathematics, Monte Carlo method is also of great significance. In the paper[7], authors employed the multilevel Monte Carlo finite element method to solve the stochastic Cahn–Hilliard–Cook equation. Monte Carlo method is also used to calculate definite integral[8].The advantages of using Monte Carlo method to approximate the definite integral is that it can estimate the value of any definite integral ,and the estimated error is independent of the dimension of the integral and the complexity of the integral region.

Monte Carlo method solves mathematical problems by constructing random Numbers that conform to certain rules. Monte Carlo method is an effective method for solving problems that are difficult to get analytical solutions or have no analytical solutions because of the complexity of calculation. Monte Carlo method combines random phenomena with certain values and solves mathematical problems effectively with statistical ideas.

2. Applications of Monte Carlo method

2.1Calculation of e by Monte Carlo method

Montmort’s problem[9, 10] is a classic problem; it was named after the mathematician Pierre Rémond de Montmort and was published in 1708.

Montmort’s problem[10]：From the top of shuffled deck of n cards having face values $1,2,\cdots,n$, cards are drawn one at a time. A match occurs if the face value on a card coincides with the order in which it is drawn. For instance, if the face values of a five-card deck appear in the order 2 5 3 4 1, then there are two matches (3 and 4).What is the probability that no match occurs?

Let A : {Matches occur}, p : probability of A. We already know that the probability limit of no match occurs is $\frac{1}{e}$. That is to say,

$$\lim_{m\to\infty} p = 1-\frac{1}{e}. \qquad (1)$$

In[1,2],The strong laws of large numbers: Let $X_1, X_2,...$ be i.i.d. random variables with $E|X_i|<\infty$.Let $EX_i=\mu$ and $S_n = X_1+\cdots+X_n$ .Then $\frac{S_n}{n}\to\mu$ a.s. as $n\to\infty$.If X_i is Bernoulli experiment, S_n is the times of successful experiments in n-Bernoulli experiments, the frequency $\frac{S_n}{n}$ of success meets

$$\frac{S_n}{n} \xrightarrow{a.s.} p \ , \ n \to \infty. \qquad (2)$$

That is to say, with probability one $\frac{S_n}{n}$ converges to p .

Then it follows from (1) and (2) that

$$e \approx \frac{1}{1 - \frac{S_n}{n}}.$$

By generating random numbers for random simulation, we getwhen $n = 10000$,

Table1 Simulations of e

m	$m_1 = 100000$	$m_2 = 200000$	$m_3 = 500000$	$m_4 = 1000000$
e	2.719682341	2.718056046	2.718277699	2.718287636
t	49.85s	118.38s	288.45s	610s

We can simulate the approximate e by Monte Carlo method. Only partial simulation results are given in Table1. Of course, Monte Carlo method needs a large number of experiments, the large n and m, the more accurate the random simulation results.

2.2Monte Carlo method and $\Phi(x)$

Monte Carlo method is also used to calculate definite integral, now what we're interested in is how we use Monte Carlo method to calculate the relevant definite integral in statistics. In probability theory, the standard normal distribution plays a fundamental role, and the three major distributions in mathematical statistics--chi-square distribution、t-distribution and F-distribution can be constructed from the standard normal distribution. So what we're concerned about now is how to calculate cumulative distribution function of the standard normal distribution, that is $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

In[2], only part $\Phi(x)$ are given in the table of the normal distribution. And these values are approximate by numerical computation. Now, we use Monte Carlo method to calculate $\Phi(x)$.

First of all, let $I_m = \int_0^m \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ .Then,take $x = 1$ as an example, we have

$$\Phi(1) = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \int_{0}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2} + I_1.$$

I_1 is the area of the curved trapezoidbelow curve $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ in $D_1 = [0,1] \times [0, \frac{1}{\sqrt{2\pi}}]$ as shown in Figure1.

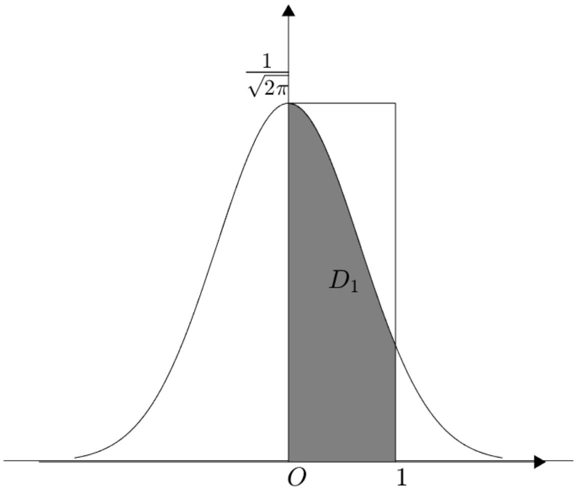

Figure 1: Graphical representation of I_1

We assume random points (a,b) obey uniform distribution in D_1, and randomly produce points in D_1.So, the probability of the point (a,b) falling into the shaded part is

$$P = \frac{I_1}{\mu(D_1)}.$$

Repeat this experiment independently for n times, and the number of times that random points falls into the shadow part is μ_n.According to the strong laws of large numbers [1],

$$\frac{\mu_n}{n} \xrightarrow{a.s.} P\ ,\ n \to \infty.$$

That is to say, with probability one $\frac{u_n}{n}$ converges to $\frac{I_1}{\mu(D_1)}$.

We simulate n test values of (a,b) with random numbers (a_i, b_i) that obey uniform distribution, and record the number of times $b_i \le f(a_i)$ occurs, then

$$\frac{u_n}{n} \approx P = \frac{I_1}{\mu(D_1)},$$

$$I_1 \approx \mu(D_1) \cdot \frac{u_n}{n} = \frac{1}{\sqrt{2\pi}} \frac{u_n}{n}. \qquad (3)$$

When $x = m$ and $m > 0$, then it follows from (3) that we have

$$I_m \approx \mu(D_m) \cdot \frac{u_n}{n} = \frac{m}{\sqrt{2\pi}} \frac{u_n}{n},$$

of which $D_m = [0,1] \times [0, \frac{m}{\sqrt{2\pi}}]$.

2.3Simulated results of $\Phi(x)$

By generating random numbers for random simulation, we get the values of $\Phi(x)$, and partial results are shown in Table2.

Table 2 Simulations of $\Phi(x)$

x	Original value	Simulated result	x	Original value	Simulated result
0.00	0.5000	0.5000	1.10	0.8643	0.8643433529
0.10	0.5398	0.5398271499	1.11	0.8665	0.8665443378
0.20	0.5793	0.5792611342	1.12	0.8686	0.8686527378
0.30	0.6179	0.6179145400	1.13	0.8708	0.8708985933
0.40	0.6554	0.6554310402	1.14	0.8729	0.8728514273
0.50	0.6915	0.6914726666	1.15	0.8749	0.8749563765
0.60	0.7257	0.7257578540	1.16	0.8770	0.8769581703
0.70	0.7580	0.7580924729	1.17	0.8790	0.8789405496
0.80	0.7881	0.7881452856	1.18	0.8810	0.8809687251
0.90	0.8159	0.8159905791	1.19	0.8830	0.8830140758
1.00	0.8413	0.8412773281	1.20	0.8849	0.8850334131
2.00	0.9772	0.9771985588	1.21	0.8869	0.8868786301
3.00	0.9986	0.9985633541	1.22	0.8888	0.8888206528

The procedures are as follows:

```
m=input ('Please enter the upper limit of the integral :')
a=unifrnd (0, m, 1, 10000000);
b=unifrnd (0, 1/ (sqrt (2*pi)), 1, 10000000);
c=normpdf (a, 0, 1);
b<=c;
P=m/ (sqrt (2*pi))*sum (ans)/10000000;
P=vpa (P+1/2, 10)
```

3. Conclusions

In this paper, first of all, on the basis of previous studies on Montmort's problem, we use Monte Carlo method to approximate the value of e.Then we focus on how to use Monte Carlo method to calculate the related definite integral in statistics. We use Monte Carlo method to calculate the cumulative distribution function of standard normal distribution, and compare it with the original value. It is found that the results of random simulation are ideal, and the running speed of computer is fast.

In theory, Monte Carlo method needs a lot of experiments. The more times the experiment is, the more accurate the results are. With the help of computer technology, Monte Carlo method realizes two advantages:

(1) Simple and fast, saves the complicated mathematical derivation and calculus process, so that ordinary people can also understand. Simple and fast, it is the basis of Monte Carlo method in today's application.

(2) The error of the modal is independent of the dimension of the problem.

References

[1]A.N. Shiryaev, (2004)*Probability second edition*, Springer-Verlag, 45-55

[2]Rick Durrett, (2007) Probability 3rd ed, Springer, 55-69

[3]George-Louis Leclerc de Buffon,(1777)*Essaid'arithmetique morale*,

[4]Soubhik Chakraborty, S.Natarajan☹(1998) Buffon’s needle problem revisited, Resonance, Vol.3 (9), pp.70-73

[5]Chhavi Agarwal, AmolMhatre, SabyasachiPatra, SanhitaChaudhury, A.Goswami (2019) Algebraic reconstruction technique combined with Monte Carlo method for weight matrix calculation in gamma ray transmission tomography, SN Applied Sciences, Vol.1 (10), pp.1-9

[6] A.K.Murtazaev, Zh.G.Ibaev (2019) Monte Carlo Calculations of the Density of States for a Two-Dimensional Anisotropic lsing Model with Competing Interactions, Bulletin of the Russian Academy of Sciences: Physics, Vol.83 (7), pp.847-849

[7]AmirrezaKhodadion , Maryam Parvizi, Mostafa Abbaszadeh, Mehdi Dehghan, Clemens Heitzinger(2019) A multilevel Monte Carlo finite element method for the stochastic Cahn–Hilliard–Cook equation, Computational Mechanics, Vol.64 (4), pp.937-949

[8]Haifeng Ma. Research Based on the Monte-Carlo method to calculate the definite integral [A]. Intelligent Information Technology Application Association. Proceedings of 2011 Third Pacific-Asia Conference on Circuits, Communications and System(PACCS 2011 V1)[C].Intelligent Information Technology Application Association (2011) Intelligent Information Technology Application Association,3.

[9]Ion Saliu (2010) *Probability Theory Live*, Xlibris, American，pp.22-25

[10]San Luis Obispo (2003) the Poisson Variation of Montmort Matching Problem