Existence Solutions of Quasilinear Delay Differential Equations via Measures of Noncompactness

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Abstract:
We study the existence of solutions for quasilinear delay differential equations with nonlocalproblem in Banach spaces. The results are established by using Darbo-Sadovskii's fixedpoint theorem.

Keywords: nonlocal conditions; noncompact measures, Darbo-Sadovskii's fixed point technique

1. INTRODUCTION

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi [7] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. El-Sayed [9] proves the existence theorem of monotonic solutions for a nonlinear functional integral equation of convolution type by Hasusdorff measure of noncompactness. The notion of "nonlocal condition" has been introduced to extend the study of the classical initial value problems; see e.g. [4, 5, 10, 14]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problems was initiated by Byszewski, we refer to some of the papers below. Byszewski[4], Byszewski and Lakshmikantham [5] give the existence

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and uniqueness of mild solutions and classical solutions when f and g satisfy the Lipschitz -type conditions. Fan et al [10] discussed semilinear differential equations with nonlocal condition using measure of noncompactness. Paul Samuel and Lisso [12] give the existence results for quasilinear delay integrodifferential equations with nonlocal conditions via measures of noncompactness. The problem of existence of solutions of quasilinear evolution equations in Banach space has been studied by several authors [6, 8, 13].
Pazy [13] considered the following quasilinear equation with local condition of the form

$$u'(t) + A(t,u)u(t) = 0, \qquad 0 < t < T$$
$$u(0) = u_0,$$

and discussed the mild and classical solutions by using the fixed point argument.
In this paper, we study the existence of mild solutions of quasilinear delay integrodifferential equations with nonlocal condition of the form

$$u'(t) + A(t,u)u(t) = f\left(t, u\big(\alpha(t)\big)\right), \quad t\epsilon[0,T] \qquad (1)$$
$$u(0) + g(u) = u_0, \qquad (2)$$

Where $A: [0,T] \times X \to X$ are continuous functions inBanach space X,
$u_0 \epsilon X, f: [0,T] \times X \to X, \; g: C([0,T];X) \to X$ and α,β are given functions.
Here $\Delta = \{t, s, \; 0 \le s \le t \le T\}$.
The results obtained in this paper generalize the results of [7, 8, 12].

2. PRELIMINARIES

Let X be a Banach space with norm $|| \cdot ||$. Let $C([0,T];X)$ be the space of X-valued continuous functions on $[0,T]$ with the norm $\big||u|\big| = \sup\{\big||u(t)|\big|, t \epsilon [0,T]\}$for u$\epsilon$ $C([0,T];X)$, and denoted $L(0,T;X)$ by the space of X-valued Bochner integrable functions on $[0,T]$ with the norm $||u||_L = \int_0^T \big||u(t)|\big| dt$. The Hausdorff's measure of noncompactness χY is defined by χ(B) = inf{$r > 0, B$ can be covered by finite number of balls with radii r} for bounded set B in a Banach space Y .

Lemma 2.1 [4]. Let Y be a real Banach space and$B, E \subseteq Y$ be bounded, with the following properties:
(i) B is precompact if and only if $\chi_X(B) = 0$.

(ii) $\chi_Y(B) = \chi_Y(\bar{B}) = \chi_Y(conv\,B)$, where B and y $conv\,B$ means the closure and convex hull of B respectively.

(iii) χ_Y (B) ≤ χ_Y (E),where $B \subseteq E$.

(iv) χ_Y (B + E)) ≤ χ_Y (B) + χ_Y (E), where $B + E = \{x + y : x \epsilon B, y \epsilon E\}$

(v) $\chi_Y(B \cup E) \le \max\{\chi_Y(B), \chi_Y(E)\}$.

(vi)$\chi_Y(\lambda B) \le |\lambda| \chi_Y(B) for\ any\ \lambda \in R$

(vii) If the map $F: D(F) \subseteq Y \to Z$ is Lipchitz continuous with constant k the $\chi_Y(FB) \le k\chi_Y(B)$ for any bounded subset $B \subseteq D(F)$, where Z is Banach space.
(viii) $\chi_Y(B) = \inf\{d_Y(B, E); E \subseteq Y$is finite valued, where $d_Y(B,E)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and E in Y.
(viii) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $\lim_{n\to\infty} \chi_Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

Lemma 2.2 (Darbo-Sadovskii [4]). If W$\subseteq$ Y is bounded closed and convex, the continuous map $F : W \to W$is a χ_Y contraction, then the map F has atleast one fixed point in W.In this we denote by χ the Hausdorff'smeasure of noncompactness of X and denote _c by the Hausdorff'smeasure of noncompactness of $C([a,T]; X)$.
To discuss the existence, we need the following Lemmas in this paper.

Lemma 2.3. [4]. If $W \subseteq C([0,T]; X)$ is bounded. Then $\chi(W(t)) \le \chi_c(W)$ for all $t \in [0,T]$, where $W(t) = \{u(t);\ u \in W\} \subseteq X$. Furthermore if W isequicontinuous on $[a,T]$, then $\chi(W(t))$ is continuous
on$[a,T]$ and $\chi_c(W) = sup\{\chi(W(t)), t \in [a,T]\}$.

Lemma 2.4. [13]. If $\{u_n\}_{n=1}^{\infty} \subset L^1(a,T;X)$ is uniformlyintegrable, then the function $\chi(\{u_n\}_{n=1}^{\infty})$ is measurable and

$$\chi(\{\int_0^t u_n(s)ds\}) \le 2\int_0^t \chi\{u_n\}_{n=1}^{\infty} ds) \qquad (3)$$

Lemma 2.5. [4]. If $W \subseteq C([0,T]; X)$ is bounded and equicontinuous, then then $\chi(W(t))$ is continuous and

$$\chi\left(\{\int_0^t W(s)ds\}\right) \le \int_0^t \chi\, \mathrm{W}(s)ds \qquad (4)$$

for all $t \in [0,T]$, where $\int_0^t$ W$(s)ds$)= $\{\int_0^t$ u$(s)ds; u \in W\}$.

The C_0 - semigroup $U_u(t,s)$ is said to be equicontinuous for $t > 0$ for all bounded set B in X.The following lemma is obvious.

Lemma 2.6. If the evolution family $\{U_u(t,s)\}_{0\le s\le T}$is equicontinuous and $\eta \in L(0,T;R^+)$, then the set $\{\int_0^t U_u(t,s)u(s)ds\}, ||u(s)|| \le \eta(s)$ for a.e s $\in [0,T]$ is equicontinuous for t $\in [0,T]$ .From [8], we know that for any fixed $\in C([0,T]; X)$, there exist a unique continuous function $U_u: [0,T]$ x$[0,T] \to B(X)$ defined on $[0,T]$x$[0,T]$ such that

$$U_u(t,s) = \int_s^t A_u U_u(\mathrm{w}, \mathrm{s})dw \qquad (5)$$

where$B(X)$ denote the Banach space of bounded linear operators from X to X with the norm $||F|| = \sup\{||Fu||: ||u|| = 1\}$, and I stands for the identity operator on $X, A_u(t) = A\big(t, u(t)\big)$.We have $U_u(t,t) = I$, $U_u(t,s)U_u(s,r) = U_u(t,r)$, where $(t,s,r) \in [0,T]$ x $[0,T]$x$[0,T]$,
$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s)$for almost all $\in [0,T]$.

For a mild solution of (1) - (2) we mean a function$u \in C([0,T];X)$ and $u_0 \in X$ satisfying the integral equation

$$u(t) = U_u(t,0)\, u_0 - U_u(t,0)g(u) + \int_0^t U_u(t,s)[f(s,u(\alpha(s))]ds, \quad (6)$$

3. EXISTENCE OF RESULTFOR LIPSCHITZ$_h$

We obtained the existence results when h is compact but without the compactness of $U_u(t,s)_{0\le s\le t\le T}$or $f$and g. In this section, wediscuss the equation $(1)-(2)$ when h is Lipschitz and$f$ and g are not Lipschitz. We assume the following assumptions:

(H1) The evolution family $\{U_u(t,s)\}_{0\le s\le t\le T}$ generated by $A(t,u)$ is equicontinuous, and $||U_u(t,s)|| \le M_0$for almost $t,s \in [0,T]$.

(H2)(a) $f : [0,T]\times X \to X$ satisfies the Caratheodorytypeconditions and there exist $m_1 \in L[0,T]$ and $b_1 \ge 0$ such that

$$|f(t,u)| \le m_1(t)b_1||u||, t \text{ a. e in } [0,T], u \in R^+$$

(b) There exist $\eta \in L(0,T;R^+), \zeta \in L(0,T;R^+)$, such that $\zeta(f(t,D) \le \eta(\mathrm{t})\,\chi(D)$ for a.e $t \in [0,T]$, and for any bounded subset $D \subset C([0,T],X)$, here we let $\eta(\mathrm{t}) \le K_0$.

(H3) (a) $g : C([0,T];X) \to X$ is continuous and compact.
(b) There exist $N > 0$ such that $||g(u)|| \le N_0$ for all $u \in C([0,T];X)$.

(H4) $\alpha: [0,T] \to [0,T]$ is non-decreasing and there exist positive constant δ_1 such that $\alpha'(t) \ge \delta_1$for $t \in [0,T]$.

$(H5)$ $h$is a Lipschitz continuous in X, there exist aconstant $L_0 > 0$ such that

$$\left|\left|h(u) - h(v)\right|\right| \le L_0\left|\left|u - v\right|\right|, u,v \in C([0,T];X).$$

Theorem: 3.1 Suppose that the assumptions $(H1)-(H6)$ are satisfied, then the equation$(1)-(2)$ has at least one mildsolution provided that

$$\left(M_0L_0 + TM_0K_0\left[\frac{1}{\delta_1}\right]\right) < 1. \quad (11)$$

Proof. Consider the map $F_1, F_2 : C([0,B];X) \to C([0,B];X)$defined by$F_1 + F_2 = F$, where

$F_1(u)(t) = U_u(t,0)h(u)$.
$F_2(u)(t) = \int_0^t U_u(t,s)\, [f\left(s,u\left(\alpha(s)\right)\right)]ds$,for $u \in C([0,B];X)$.We define
$W_0 = \{u \in C([0,B];X : ||u(t)|| \le \Omega(t)$ for all $t \in [0,T]$ and let $W = \overline{conv}FW_0$.
We know that W is abounded closed convex and equicontinuous subset of$C([0,T];X)$ and $FW \subset W$. We shall prove that F isχ_c-contraction on W. Then Darbo-Sadovskii's fixedpoint theorem can be used to get a fixed point of Fin$W$, which is a mild solution of $(1)-(2)$. First, for everybounded subset $B \subset W$, from the $(H7)$ and Lemma 2.1, we have

$\chi_c(F_1 B) = \chi_c\big(U_B(t,0)g(B)\big)$
$\leq M_0 \chi_c(g(B))$
$\leq M_0 L_0 \chi_c(B)$ (12)
Next, for every bounded subset $B \subset W$, for $t \in [0,T]$and every ϵ> 0, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset B$, such that
$\chi(F_2 B(t)) \leq 2\chi\{F_2 u_k(t)\}_{k=1}^{\infty} + \epsilon$.

Note that B and $F_2 B$ are equicontinuous, we can getfrom Lemma 2.1, Lemma 2.4, Lemma 2.5 and $(H2)$ and $(H4)$
that

$$\chi(F_2 B(t)) \leq \chi\left(\int_0^t U_u(t,s)[f(s,\{u_k(\alpha(s))\}_{k=1}^{\infty})]ds\right)$$

$\leq 2M_0 \int_0^t \left[\chi\left(f\left(s,\{u_k(\alpha(s))\}_{k=1}^{\infty}\right)\right)\right] ds$
$\leq \left(2M_0 K_0 \left[\frac{1}{\delta_1}\right]\right)\chi_c(B)T + \epsilon$for all $t \in [0,T]$.

Since ϵ > 0 is arbitrary, we have
$\chi(F_2 B(t)) \leq T\left(2M_0 K_0 \left[\frac{1}{\delta_1}\right]\right)\chi_c(B)$ (13)
For any bounded $B \subset W$.

Now, for any subset $B \subset W$, due to Lemma 2.1,(12) and (13) we have
$\chi_c(FB) \leq \chi_c(F_1 B) + \chi_c(F_2 B)$

$\chi_c(FB) \leq \left[M_0 L_0 + T\left(2M_0 K_0 \left[\frac{1}{\delta_1}\right]\right)\chi_c(B)\right]$.

We know that F is aχ_c-contraction on W. By Lemma2.2, there is a fixed point u of F in W, which is asolution of $(1) - (2)$. This completes the proof.

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