A NOTE ON PERFECTLY CONTINUOUS FUNCTIONS

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ABSTRACT
The concept of perfectly continuous functions was introduced by Noiri [4]. In this paper some more properties of perfectly continuous functions are obtained.

INTRODUCTION
In 1984, Noiri [4] introduced the class of perfectly continuous functions and investigate its relationship with other strong forms of continuity such as complete continuity [1], strong continuity [2]. In this paper we give new results on perfectly continuous functions.

Definition 1. A function \( f : X \to Y \) is said to be perfectly continuous [4] if the inverse image of every open set in \( Y \) is both open and closed in \( X \).

Definition 2. A function \( f : X \to Y \) is said to be strongly continuous [2] iff the inverse image of every open subset in \( Y \) is open and closed in \( X \).

Definition 3. A function \( f : X \to Y \) is said to be completely continuous [1] if the inverse image of every open set in \( Y \) is regular open in \( X \), where \( A \) is regular open iff \( A = \text{Int}(\text{cl}A) \).

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Noiri proved that:
Strong continuity $\Rightarrow$ Perfect continuity $\Rightarrow$ Complete continuity $\Rightarrow$ continuous.
He proved also by means of examples that the converse implications are not true in general.

**Theorem 1.** Every perfectly continuous function into a $T_1$ space is strongly continuous.

**Proof:** Let $f : X \to Y$ be perfectly continuous and $Y$ is a $T_1$ space. Let $A$ be any subset of $Y$. Then $f^{-1}(A)$ is open and closed in $X$, since singletons are closed in a $T_1$-space.

**Theorem 2.** If $f : X \to Y$ is a perfectly continuous function from a connected space $X$ on to any space $Y$, then $Y$ is an indiscrete space.

**Proof:** If possible, suppose that $Y$ is not indiscrete. Let $A$ be a proper non-empty open subset of $Y$. Then $f^{-1}(A)$ is a proper non-empty clopen subset of $X$, which is a contradiction to the fact that $X$ is connected.

**Theorem 3.** Every perfectly continuous function from an extremally disconnected space is perfectly continuous.

**Proof:** The proof is obvious since every regular open subset in an extremally disconnected space is clopen.

**Theorem 4.** Let $f : X \to Y$ be perfectly continuous and one-to-one. If $Y$ is $T_0$, then $X$ is Urysohn.

**Proof:** Let $a$ and $b$ be any pair of distinct points of $X$. Then $f(a) \neq f(b)$. Since $Y$ is $T_0$, there exists an open set $U$ containing one of them say $f(a)$ but not $f(b)$. Then $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$ is clopen. $f$ is perfectly continuous implies $f^{-1}(U)$ is clopen. Also, $a \in f^{-1}(U)$ and $b \in X - f^{-1}(U)$. Thus $a$ and $b$ are separated by disjoint closed sets. Hence $X$ is Urysohn.

**Theorem 5.** If $X$ is connected and $Y$ is $T_0$, then the only perfectly continuous functions $f : X \to Y$ are constant functions.

**Proof:** If possible, let $f(X)$ consist of more than one point. But then $f(X)$ is $T_0$ also, which is a contradiction.

**Definition 4.** A function $f : X \to Y$ is said to be almost open [3] if the image of every regular open set is open.

**Theorem 6.** The image of a locally connected space under a perfectly continuous almost open function is locally connected.

**Proof:** Let $f : X \to Y$ be perfectly continuous and almost open. Let $X$ be locally connected. Let $U$ be any open subset of $Y$ and $f(x) \in U$. Then $x \in f^{-1}(U)$. Now, $f^{-1}(U)$ is open and closed in $X$. Since $X$ is locally connected, there exists a connected open subset $V$ of $X$ such the
\(x \in V \subseteq f^{-1}(V).\) But \(f^{-1}(U)\) is closed, therefore, \(x \in V \subseteq \text{cl}(V) \subseteq f^{-1}(U).\) Hence \(V\) and \(\text{cl}(V)\) are connected. Also, \(\text{int}(\text{cl}(V))\) is connected. Clearly, \(x \in \text{int}(\text{cl}(V)) \subseteq f^{-1}(U).\) Hence \(f(x) \in f(\text{int}(\text{cl}(V))) \subseteq U.\) Now, \(f(\text{int}(\text{cl}(V)))\) is connected and \(f(\text{int}(\text{cl}(V)))\) is open since \(f\) is open. Hence \(Y\) is locally connected.

**Theorem 7.** If \(f : X \to Y\) is perfectly continuous and \(A \subseteq X,\) then \(f|_A : A \to Y\) is perfectly continuous.

**Proof:** Let \(V\) be any open subset of \(Y.\) Then \(f^{-1}(V)\) is clopen in \(X.\) Then \(f^{-1}(V) \cap A = \left(f|_A^{-1}\right)(V)\) is clopen in \(A.\) Hence \(f|_A\) is perfectly continuous.

**REFERENCES**


