DOUBLE FINITE FOURIER SINE TRANSFORM AND COMPUTER SIMULATION FOR BIHARMONIC EQUATION OF PLATE DEFLECTION

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Published: 13 April 2019

Abstract
We will present the biharmonic equation that models the deflection u = u (x, y) of aRectangular Elastic Plate, which will be subject to a concentrated load. The solution of this equation will be established, using the Double Finite Fourier Sine Transform, solution that will be simulated in a computer using the Computer Algebra System (CAS) Mathematica

1. INTRODUCTION. We develop the application method of the Double Finite Fourier Sine Transform, to find an explicit solution to thebiharmonic equation, that models the deflection u = u (x, y), of a Simple Supported Rectangular Elastic Plate, where,on the edge of the plate the deflection and bending moments are zero, what will be indicated in the auxiliary conditions. It is considered the problem of a concentrated load applied at a point (ε, μ) inside the plate, which we express using the Dirac Delta as:$P\ \delta_{\varepsilon}(x)\delta_{\mu}(y)$, where P is a constant.

The constant flexural rigidity of the plate is given by:$D = \frac{2}{3}\frac{Eh^3}{(1-\sigma^2)}$, where:

E = Young's module, $\sigma =$ Poisson radio and h = elastic thickness.
Let's start with the definition of the transformed.

Cite this article: Pasquel, F. (2019). Double Finite Fourier Sine Transform and Computer Simulation for Biharmonic Equation of Plate Deflection. *European International Journal of Science and Technology*, 8(3), 59-64.

DEFINITION 1.1. We define the Double Finite Fourier Sine Transform of a function F (notation: $\mathcal{DF}_{is}$), con domain over a rectangular region: $0 \leq x \leq a,\ 0 \leq y \leq b$, as the function f_s:

$$\mathcal{DF}_{i\,s}(F(x,y)) = f_s(m,n) = \int_0^a \int_0^b F(x,y)\ Sin\left(\frac{m\,\pi\,x}{a}\right)\ Sin\left(\frac{n\,\pi\,y}{b}\right)\ dy\ dx$$

The Inverse Double Finite Fourier Sine Transform of the function f_s(notation:$\mathcal{DF}_{is}{}^{-1}$) will be:

$$\mathcal{DF}_{is}{}^{-1}\big(f_s(m,n)\big) = F(x,y) = \left(\frac{4}{ab}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_s(m,n)\,Sin\left(\frac{m\pi x}{a}\right)Sin\left(\frac{n\pi y}{b}\right)$$

the following theorem [2] will be very important:

THEOREM 1.2. If the function F, vanishes on the boundary of the rectangular region:
$0 \leq x \leq a,\ 0 \leq y \leq b$ then:

$$\mathcal{DF}_{is}\left(\frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2}\right) = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)f_s(m,n)$$

We have now, thebiharmonic equation, with the indicated auxiliary conditions ($\delta\ \ (x)$, is the Dirac delta)

$$\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{P}{D}\,\delta_\varepsilon(x)\delta_\mu(y)$$

$$\begin{cases} u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0 \\ u_{xx}(0,y) = u_{xx}(a,y) = 0 \\ u_{yy}(x,0) = u_{yy}(x,b) = 0 \\ 0 < x < a,\ \ 0 < y < b\ , \end{cases}$$

We will now apply the Double Finite Fourier Sine Transform double to thebiharmonic equation:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = \frac{P}{D}\mathcal{DF}_{i\,s}\left(\,\delta_\varepsilon(x)\delta_\mu(y)\,\right)$$

where

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^2 u}{\partial x^2}\right) = -\pi^2\frac{m^2}{a^2}\mathcal{DF}_{i\,s}\big(u(x,y)\big) = -\pi^2\frac{m^2}{a^2}\mathbb{U}_s(m,n)$$

Where, we use the notation:

$$\mathcal{DF}_{i\,s}\big(u(x,y)\big) = \mathbb{U}_s(m,n)$$

Then:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4}\right) = \mathcal{DF}_{i\,s}\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial x^2}\right)\right) = -\pi^2\frac{m^2}{a^2}\mathcal{DF}_{i\,s}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4}\right) = \left(-\pi^2\frac{m^2}{a^2}\right)\left(-\pi^2\frac{m^2}{a^2}\right)\mathcal{DF}_{i\,s}\big(u(x,y)\big)$$

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4}\right) = \pi^4\frac{m^4}{a^4}\mathbb{U}_s(m,n)$$

in analogous form it gets:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial y^4}\right) = \pi^4 \frac{n^4}{b^4}\mathbb{U}_s(m,n)$$

For the application of the transform to the mixed derivative, we apply the Theorem 1.2.:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^2\,\partial y^2}\right) = \mathcal{DF}_{i\,s}\left(\frac{\partial^2}{\partial x^2}(\frac{\partial^2 u}{\partial y^2})\right) = \left(-\pi^2\frac{m^2}{a^2}\right)\left(-\pi^2\frac{n^2}{b^2}\right)\mathcal{DF}_{i\,s}\big(u(x,y)\big)$$

Then:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^2\,\partial y^2}\right) = \pi^4\frac{m^2\,n^2}{a^2 b^2}\mathbb{U}_s(m,n)$$

Then, the application of the Double Finite Fourier Sine Transform, to thebiharmonic equation, will be:

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = (\,\pi^4\frac{m^4}{a^4} + 2\,\pi^4\frac{m^2\,n^2}{a^2 b^2} + \pi^4\frac{n^4}{b^4}\,)\ \mathbb{U}_s(m,n)$$

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = \pi^4(\,\frac{m^2}{a^2} + \frac{n^2}{b^2}\,)^2\ \mathbb{U}_s(m,n)$$

We now apply the Double Transform to the Dirac delta:

$$\mathcal{DF}_{is}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,) = \int_0^a\int_0^b \delta_\varepsilon(x)\delta_\mu(y)\,Sin\left(\frac{m\pi x}{a}\right)Sin\left(\frac{n\pi y}{b}\right)dydx)$$

$$\mathcal{DF}_{i\,s}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,) = \int_0^a(\,\delta_\varepsilon(x)\,Sin\,\left(\frac{m\,\pi\,x}{a}\right)\int_0^b\delta_\mu(y)\,Sin\,\left(\frac{n\,\pi\,y}{b}\right)dy)\,dx$$

$$\mathcal{DF}_{i\,s}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,) = \int_0^a\delta_\varepsilon(x)\,Sin\,\left(\frac{m\,\pi\,x}{a}\right)\,Sin\,\left(\frac{n\,\pi\,\mu}{b}\right)\,dx$$

$$\mathcal{DF}_{i\,s}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,) = Sin\,\left(\frac{n\,\pi\,\mu}{b}\right)\int_0^a\delta_\varepsilon(x)\,Sin\,\left(\frac{m\,\pi\,x}{a}\right)\,dx$$

Then:

$$\mathcal{DF}_{i\,s}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,) = Sin\,\left(\frac{m\,\pi\,\varepsilon}{a}\right)\ Sin\,\left(\frac{n\,\pi\,\mu}{b}\right)$$

$$\mathcal{DF}_{i\,s}\left(\frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) = \frac{P}{D}\mathcal{DF}_{i\,s}\ (\,\delta_\varepsilon(x)\delta_\mu(y)\,)$$

$$\pi^4(\,\frac{m^2}{a^2} + \frac{n^2}{b^2}\,)^2\mathbb{U}_s(m,n) = \frac{P}{D}\ Sin\,\left(\frac{m\,\pi\,\varepsilon}{a}\right)\ Sin\,\left(\frac{n\,\pi\,\mu}{b}\right)$$

Then:

$$\mathbb{U}_s(m,n) = \frac{P}{D\,\pi^4 \beta^4} Sin\left(\frac{m\,\pi\,\varepsilon}{a}\right)\; Sin\left(\frac{n\,\pi\,\mu}{b}\right)$$

(we use the notation : $\beta^2 = (\frac{m^2}{a^2} + \frac{n^2}{b^2})$)

Now applying the Inverse Double Finite Fourier Sine Transform to the function $\mathbb{U}_s$:

$$\mathcal{DF}_{i\,s}{}^{-1}\big(\mathbb{U}_s(m,n)\big) = u(x,y)$$

Finally, we arrive at the solution of the deflection equation that we place to perform our computational simulation:

$$u(x,y) = \frac{4\,P}{D\;a\;b\;\pi^4} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\beta^4}\; Sin\left(\frac{m\,\pi\,\varepsilon}{a}\right)\; Sin\left(\frac{n\,\pi\,\mu}{b}\right)\right) Sin\left(\frac{m\,\pi\,x}{a}\right)\; Sin\left(\frac{n\,\pi\,y}{b}\right)$$

2. COMPUTER SIMULATION FOR BIHARMONIC EQUATION

The Mathematica software has been used to generate a video of the deflection of the plate, of dimensions a and b, by the action of a concentrated load, in the point (ε, μ).

The video shows how the deflection progresses, as the concentrated load increases.
Arbitrary data has been used.

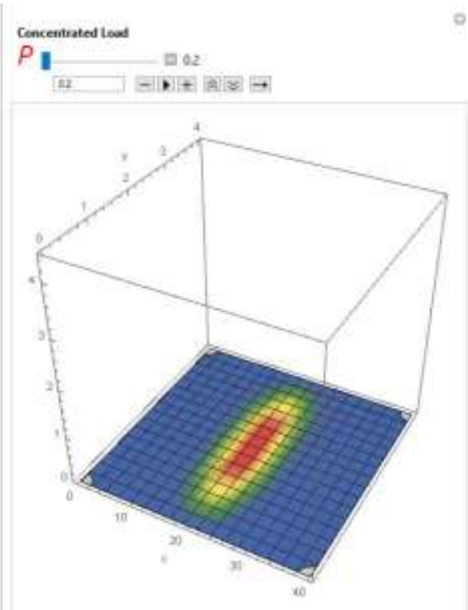
Data

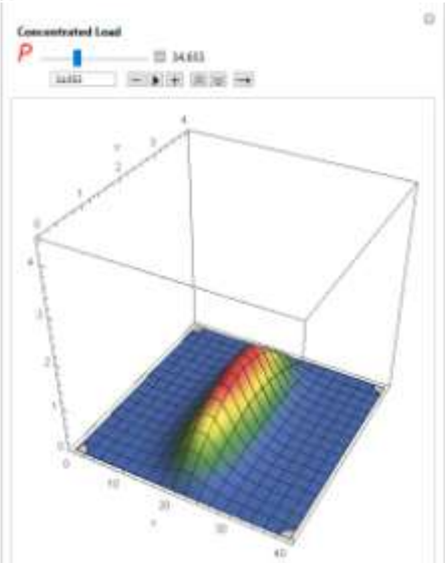
```
Clear["Global`*"];
d = 10^3;
a = 40;
b = 4;
ε = 20;
μ = 2;
```

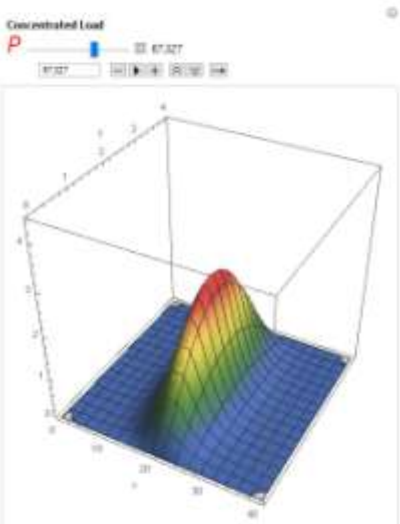
Approximate solution

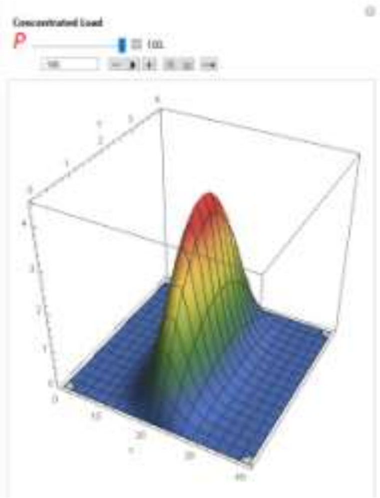
$$\texttt{u[x\_, y\_, p\_]} = \frac{4\,p}{d\,a\,b\,\pi^4} \sum_{m=1}^{20}\sum_{n=1}^{20}\left(\frac{1}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^4}\, \mathrm{Sin}\left[\frac{m\,\pi\,\varepsilon}{a}\right]\, \mathrm{Sin}\left[\frac{n\,\pi\,\mu}{b}\right]\right) \mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]\, \mathrm{Sin}\left[\frac{n\,\pi\,y}{b}\right];$$

```
Manipulate[Plot3D[Evaluate[u[x, y, p]], {x, 0, a}, {y, 0, b}, PlotPoints → 20, AxesLabel → {"x", "y", ""},
  ColorFunction → "DarkRainbow", BoxRatios → 1, PlotRange → {0, 4.5}], Style["Concentrated Load", Bold],
 {{p, 0.2, "P"}, 0, 100, 0.001, ImageSize → Small, Appearance → "Labeled"}]
```


Concentrated Load
P
0.2


Concentrated Load
P
34.653


Concentrated Load
P



REMARK. The case of aconcentrated load at the point (ε, μ), has been described.
For the case of a load$P(x, y)$over a regionD_1 :$\sigma \leq x \leq \tau,\ \varphi \leq y \leq \omega$, inside the region D, double integration should be used; the solution now will be[2]:

$$u(x, y) = \frac{4}{D\ a\ b\ \pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [\int_{\sigma}^{\tau} \int_{\varphi}^{\omega} P(\alpha, \beta)\ \ Sin\left(\frac{m\,\pi\,\alpha}{a}\right)\ \ Sin\left(\frac{n\,\pi\,\beta}{b}\right)\ d\alpha\ d\beta\ ]\ \frac{1}{\beta^4}\ Sin\left(\frac{m\,\pi\,x}{a}\right)\ \ Sin\left(\frac{n\,\pi\,y}{b}\right)$$

Which would now be used to perform computer simulation.

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