BERNOULLI LOOPS IN SURFACE GRAVITY WAVES

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ABSTRACT
With help from Bernoulli's law applied to closed streamlines (loops) and the cross-stream force balance, the depth decay rate is calculated for the orbital speed of fluid particles in progressive surface gravity waves. It appears to be unlikely that the decay rate will turn out to be the exponential one predicted classically. Experimental data, from streak photographs, are needed to provide the orbital radius of curvature as a function of depth in order to complete the solution of the linear governing equation for the orbital speed in which the radius is a non-constant coefficient.

Keywords: Surface gravity waves, fluid particle velocity

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1. Introduction

Only recently has Bernoulli's law been applied to steady flow streamlines that are closed loops, and the particular application exhibited was solid body rotation of a fluid contained in a vertically spinning cylinder, closed at the bottom and open to the air at the top [1].

Within progressive surface gravity waves, in the fixed reference frame (fixed to the shore or solid bottom of a body of water), all fluid particles have orbital paths that are very nearly closed circles. Classically Bernoulli's law was used along the surface streamline only, at the air/water interface and in the steady reference frame (the wave shape is fixed and fluid flows by underneath it), but probably never before along the individual orbital paths of the fluid particles in these waves. A first attempt to do this is the main reason for the discussion below. Is there anything new that can be learned about the surface gravity waves that have already been studied mathematically for several centuries? Answer is: yes.

Almost 30 years ago a horizontal balance at the surface between the centrifugal force and a pressure force, called the cyclostrophic balance, immediately reproduced the classical dispersion relation of propagating surface gravity waves of sinusoidal form with very little effort [2]. That result awakened in me a greater appreciation for the potential usefulness of the centrifugal force than is usually to be found in the field of Physics.

2. Method

A progressive surface gravity wave moves past an observer in the fixed reference frame. Consider a vertical line passing through a crest or trough at one instant of time. Along this line all the tops at the crest or bottoms at the trough of the circular orbits are lined up (nested) from the surface down until vanishing at depth (greater than a about one wavelength). Bernoulli's law along each circular streamline is

$$p = const - \frac{1}{2}\rho v^2 - \rho g z \tag{1}$$

Where p is pressure, ρ constant density, v is the flow speed tangent to the streamline, g the acceleration of gravity and z is the vertical coordinate, positive upward and zero at the top of a crest. Assume for simplicity that the constant in (1) is the same for all streamlines.

Across the streamlines in the vertical direction there must be a balance of three forces in order to maintain steady motion: gravity, a pressure gradient and the centrifugal force [3]. Under a crest this balance is

$$\frac{dp}{dz} + \rho g = \frac{\rho v^2}{r} \tag{2}$$

Where r is the radius of curvature of the circular streamlines and it is a function of z.

Equations (1) and (2) are two equations in two unknowns, p and v, and there is a non-constant coefficient r. These equations have been written down before and partially solved [3] for the surface gravity wave in the steady reference frame, but here in the fixed frame the velocity variable and the radius of curvature function have very different meanings. For example, in the earlier study r here was replaced by R, the radius of curvature of a streamline in the steady frame. R has a minimum value at a crest or trough and increases to infinity at depth where the streamlines are straight, whereas r has a maximum value at the surface decreasing to zero at depth. If a purely sinusoidal wave form is assumed, then there is an inverse relation between $r$and R:

$$R = \frac{1}{rk^2} \tag{3}$$

Where k is the wave number (a constant).

Therefore, to obtain one governing equation in one unknown (velocity), eliminate the pressure by taking the z derivative of (1) and combining it with (2).

$$\frac{dv}{dz} = -\frac{v}{r} \tag{4}$$

Which is a linear first order differential equation with a non-constant coefficient. Separate variables, integrate both sides, and raise both sides to the exponent of e to get

$$\frac{v}{v_0} = e^{-\int \frac{dz}{r}} \tag{5}$$

Where v_0 is an integration constant.

To complete the solution of (5) *r(z)* must be supplied, either from observations or some other theory. Then the pressure can be found by inserting the velocity solution into (1) or (2).

3. Discussion

Even though the function *r(z)*is not known in detail a priori, (5) shows that the solution for the velocity is not sensitive to the exact trajectory r takes from its maximum value at the surface to zero at depth, because it is involved in an integration over depth in the exponent. Also it is difficult to imagine from (5) that the orbital velocity will turn out to decrease exponentially the way the classical theory predicts. Therefore, (5) is a new result that should be tested by observations in the future. Streak photographs of small neutrally buoyant particles in the laboratory could be modeled algebraically or numerically and then integrated on the computer. A few old oceanography (e.g. [4], [5]) or fluid dynamics text books contain such photographs, but newer more carefully done ones would be better. That is one suggestion.

One way to evaluate integration constants, like v_0 in (5), is to compute the total orbital flow of mass under a crest by a vertical integration and set it equal to the total mass flow under a trough.

Observations show that the fluid particle orbits of a progressive surface gravity wave are not closed circles but that there is a very small advance of each particle in the wave propagation direction after every orbital period. This advance has been called the Stokes drift, which has important consequence in other problems, such as the force on a wall when waves reflect from it. However, this fact does not affect the main conclusion obtained above.

Finding that the particle velocity very likely does not have the classical exponential depth decay rate is consistent with the earlier result that the flow speed under a crest, for example, in the steady reference frame also does not decay exponentially in all probability [2]. After all, the two reference frames, steady and fixed, are related by the phase velocity of the wave, which is a constant (i.e. independent of depth).

In the first attempt to obtain the orbital velocity depth decay rate [2] without the aid of Bernoulli's law it is apparent now, in view of the present results, that there is a flaw in that argument.

4. Conclusion

For the first time Bernoulli's law is applied to the circular fluid particle paths in a progressive surface gravity wave. Also the cross-stream force balance is used. A result is obtained for the orbital velocity as a function of depth, which predicts a decay rate down from the surface different from the classical exponential one in all probability. Missing from a complete solution at the present time is the orbital radius as a function of depth. But since the governing equation is linear, it can be completed numerically if not algebraically when the missing information becomes available.

References

[1] Kenyon, K. E. (2019) Bernoulli Loop. European Int. J. of Sci. and Tech., 8, 57-58.

[2] Kenyon, K. E. (1991) Cyclostrophic Balance in Surface Gravity waves. *J. of the Oceanogr. Soc. Japan*, 47, 45-48.

[3] Kenyon, K. E. (2013) Depth Decay Rate for Surface Gravity Wave Pressure and Velocity. Natural Science, 5, No. 1.

[4] Neuman, G. and W. J. Pierson, Jr. (1966) Prnciples of Physical Oceanography. Prentice-Hall, New Jersey, 545 pp.

[5] Wiegel, R. L. (1964) Oceanographical Engineering. Prentice-Hall, New Jersey, 532 pp.