Therefore, the new proposed algorithm, for $x \geq 50$, is:
for $n = 3$ to $n = \lfloor \frac{x}{2} \rfloor$,
for $k = 1$
to $k = \lfloor \left(2^{1/n} - 1\right) \left(\frac{x}{x+1}\right)^{(1-2^{(1/n)-1})/(2^{(1/n)}-1)} \cdot x \rfloor$,
for $y = \lceil \max \left\{ \left(\frac{x}{nk}\right)^{1/(n-1)} - \frac{k}{x}, \left[\frac{(2^{1/n}-1)x}{k}\right]^{1/(n-1)} \right\} \cdot x \rceil$
to $\lfloor \frac{-k + \sqrt{4[x^n/(kn)]^{2/(n-1)} - (k^2/3)}}{2} \rfloor$,
check whether:

$$\left(\frac{x}{y}\right)^n + 1 \quad = \quad \left(1 + \frac{k}{y}\right)^n$$

and record:

$$\gamma \quad := \quad -\frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x}.$$

5 Complexity Considerations

We would like to discuss a little bit about the complexity of the second algorithm, which works for $x \geq 50$.
If we neglect the number of additions, subtractions, multiplications, divisions, radical extractions, and rasing to power operations, and count only the number of cycles, then for a fixed x, we can say the following about our algorithm:
Since $3 \leq n \leq x/2$, $1 \leq k < (2^{1/n} - 1)x$, and $A - k < y \leq A - (k/2)$, where:

$$A \quad := \quad \left(\frac{x}{nk}\right)^{1/(n-1)} \cdot x, \tag{5.1}$$

and because of the following limits:

$$\lim_{n\to\infty} \frac{2^{1/n} - 1}{1/n} \quad = \quad \ln 2 \tag{5.2}$$

and

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n\right) \quad = \quad c, \tag{5.3}$$

where c denotes Euler's constant, the number of cycles in our algorithm grows at most like:

$$\begin{aligned}
&\sum_{n=3}^{\lfloor x/2\rfloor}\sum_{k=1}^{\lfloor(2^{1/n}-1)x\rfloor}\frac{k}{2}\\
\approx\ &\frac{1}{2}\sum_{n=3}^{\lfloor x/2\rfloor}\frac{\left(2^{1/n}-1\right)x\left[\left(2^{1/n}-1\right)x+1\right]}{2}\\
\approx\ &\frac{x}{4}\sum_{n=3}^{\lfloor x/2\rfloor}\frac{\ln 2}{n}\left(\frac{\ln 2}{n}x+1\right)\\
\approx\ &\frac{1}{4}\left[x^2(\ln 2)^2\sum_{n=3}^{\lfloor x/2\rfloor}\frac{1}{n^2}+x\ln 2\sum_{n=3}^{\lfloor x/2\rfloor}\frac{1}{n}\right]\\
\approx\ &\frac{1}{4}\left[x^2(\ln 2)^2\left(\frac{\pi^2}{6}-1-\frac{1}{2^2}\right)+x\ln 2\left(\ln\left(\frac{x}{2}\right)+c-1-\frac{1}{2}\right)\right].
\end{aligned}$$

So, we can say that our algorithm has an order of $O(x^2)$.
We would like to mention that there are sharper inequalities than the one relating the logarithmic mean and the arithmetic mean of two positive numbers, in the sense that the arithmetic mean can be replaced by smaller Hölder means, see [3], [5], and [4], but we believe that those inequalities are a little bit impractical to use, and their use do not speed up too much our algorithm. We are including below the maximum value for γ computed by both first and second algorithm. In what follows

$$\gamma := \sup\left\{\frac{\ln|1+(x/y)^n-(z/y)^n|}{\ln x} \mid 5\le x\le x_{MAX}\right\}, \qquad (5.4)$$

where x_{MAX} is the maximum value assigned to x.

6 Numerical Results

Table 1: First Algorithm

xmin =	5	5	5	5	5
xmax =	200	300	500	1000	2000
Trials =	1624697	5729625	27675910	230589149	1895272741
γ =	4.435969439	4.435969439	4.435969439	4.435969439	4.435969439

Table 2: Second Algorithm

xmin =	50	50	50	50	50
xmax =	200	300	500	1000	2000
Trials =	9154	83241	1337896	10767073	86368294
γ =	3.475021756	3.475021756	3.725872227	3.725872227	3.725872227

Important observation: the highest value of γ (4.435969439) is attained in the lower range of xmax. The algorithm was actually ran wit xmax values beyond 3000 and γ never increased.

Note: From $x = 50$ to $x = 3000$ it takes $12 - 36$ hours, depending on the machine speed.

7 Conjecture

Finally, we would like to end the paper with a conjecture, about the exponent γ suggested by the numerical algorithm from this paper, namely:

Conjecture 7.1 *Under the bounds, obtained in this paper, for* $n \in [3, N(x)] \cap \mathbb{N}$*,* $y \in [x+1, Y(x+1)] \cap \mathbb{N}$*, and* $k \in [1, K(x)] \cap \mathbb{N}$*, relative to* x*, there exists a finite positive number* $\gamma \approx 4.435969439$*, such that:*

$$\inf \left\{ \frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x} \right\} = -\gamma. \tag{7.1}$$

We hope that this conjecture, which may be even harder to prove than Fermat's Last Theorem, will make this celebrated theorem interesting for people working in the area of Analysis. Moreover, as this paper humbly shows, Fermat's Last Theorem, even though proven by Andrew Wiles, may still continue to stimulate new mathematical research.

References

[1] R. Bhatia, The logarithmic mean, *Resonance*, June 2008, Volume 13, Issue 6, pp. 583-594.

[2] P.S. Bullen, D.S. Mitrinović, and P.M.Vasić, *Means and Their Inequalities*, Mathematics and Its Applications (East European Series), D.Reidel Publishing Company, Dordrecht, The Netherlands, 1988, ISBN 90-277-2629-9.

[3] B.C. Carlson, The Logarithmic Mean, *Amer. Math. Monthly*, 79, (1972), 615-618.

[4] C.O. Imoru, The power mean and the logarithmic mean, *Internat. J. Math. Math. Sci.*, Volume 5 (1982), Issue 2, Pages 337-343.

[5] T.-P. Lin, The Power Mean and the Logarithmic Mean, *Amer. Math. Monthly*, 81 (1974), 879-883.

[6] F. Qi and S.-L. Xu, The function $(b^x - a^x)/x$: Inequalities and properties, *Proc. Amer. Math. Soc.*, **126**, (1998), no. 11, 3355-3359.