A conjecture derived from a vertical approach of Fermat’s Last Theorem

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Abstract

An algorithm for checking Fermat’s Theorem is presented first. An elementary double inequality is proven next. We use this inequality to refine our algorithm. The study evolves into a conjecture about how far from zero the difference between the sum of the n–powers of two natural numbers, and the n–power of another natural number may be, in terms of a critical exponent γ, for n ≥ 3.

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1 Introduction

Despite the fact that Andrew Wiles has proved the celebrated Fermat’s Last Theorem, using very advanced mathematical knowledge, the search for an elementary proof of this theorem continues. Even though this search may not lead to a proof of Fermat’s Last Theorem, it could still be useful for the discovery of new mathematical theorems.

The classic “horizontal approach” fixes the natural power \( n \geq 3 \), and aims to show that no triplet \((x, y, z)\) of natural numbers exists, such that:

\[
x^n + y^n = z^n. \tag{1.1}
\]

The “vertical approach” fixes the triplet \((x, y, z)\) of natural numbers, and aims to show that no natural power \( n \geq 3 \) exists, such that (1.1) holds.

In this paper, we use the vertical approach in the following way: we order the natural numbers \( x, y, z \), assuming that:

\[
x < y < z, \tag{1.2}
\]

and fix the smallest one, \( x \). Then we create a numerical algorithm to check that no \( y \) and \( z \), greater than \( x \), and no \( n \), greater than 2, natural numbers exist such that (1.1) holds. In order to make such an algorithm more and more efficient, we have to come up with finer and finer inequalities about the lower and upper bounds of the variables involved.

We want to make it clear that we do not claim to prove Fermat’s Last Theorem. Rather, we use it as a premise to introduce some interesting mathematical inequalities.

Our algorithm suggests the existence of a negative lower bound \(-\gamma\) for the expression:

\[
\frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x}. \tag{1.3}
\]

If such a lower bound exists, then Fermat’s Theorem follows, since it implies:

\[
\left| 1 + \frac{x^n}{y^n} - \frac{z^n}{y^n} \right| \geq x^{-\gamma} \tag{1.4}
\]

\[
> 0. \tag{1.5}
\]
2 A beginning algorithm

Let us assume that \( x, y, \) and \( z, \) satisfy equation (1.1), and:
\[
x < y < z.
\] (2.1)

Let’s define:
\[
k := z - y \in \mathbb{N}.
\] (2.2)

Equation (1.1) becomes:
\[
x^n + y^n = (y + k)^n
\Rightarrow x^n + y^n = y^n + nky^{n-1} + \cdots + k^n.
\]

Subtracting \( y^n \) from both sides, we can see that:
\[
x^n \geq nky^{n-1} + k^n.
\] (2.3)

From here we conclude that:
\[
x^n > nky^{n-1},
\] (2.4)

which after dividing both sides by the positive number \( x^{n-1}, \) since \( y \geq x + 1 \)
and \( k \geq 1, \) implies:
\[
x > nk \left( \frac{y}{x} \right)^{n-1}
\geq n \cdot 1 \cdot \left( \frac{x + 1}{x} \right)^{n-1}
= n \left( 1 + \frac{1}{x} \right)^{n-1}.
\]

Applying now Bernoulli inequality:
\[
(1 + t)^n \geq 1 + nt,
\] (2.5)

for \( t := 1/x > -1, \) we obtain:
\[
x > n \left( 1 + \frac{1}{x} \right)^{n-1}
\geq n \left( 1 + \frac{n - 1}{x} \right)
= n \cdot \frac{x + n - 1}{x}.
\]
Multiplying both sides by $x$, in the last inequality, we get:

$$x^2 > n(x - 1 + n),$$

which is equivalent to:

$$x^2 > (x - 1)n + n^2.\tag{2.6}$$

Since both $x^2$ and $(x - 1)n + n^2$ are integers, we must have:

$$x^2 - 1 \geq (x - 1)n + n^2.$$

Moving all terms to the right, we see that $n$ satisfies the quadratic inequality:

$$n^2 + (x - 1)n - x^2 + 1 \leq 0,$$ \tag{2.6}

and since the graph of the left hand side, viewed as a function of $n$, is an open upward parabola, we conclude that $n$ must be located in between the two roots of this quadratic polynomial (one is positive and the other negative). Thus $n$ must be less than or equal to the greater (positive) of the two roots, that means:

$$n \leq \frac{-(x - 1) + \sqrt{(x - 1)^2 - 4(-x^2 + 1)}}{2}$$

$$= \frac{1 - x + \sqrt{(x - 1)(x - 1 + 4(x + 1))}}{2}$$

$$= \frac{1 - x + \sqrt{(x - 1)(5x + 3)}}{2}.$$

Hence we obtain the following upper bound for $n$:

$$n \leq \frac{1 - x + \sqrt{(x - 1)(5x + 3)}}{2}. \tag{2.7}$$

Constraint (2.7) already shows, that for $x = 4$, there are no $n \geq 3$, $y > x$, and $z$ for which equation (1.1) holds, since in that case (2.7) becomes:

$$n \leq \frac{1 - 4 + \sqrt{(4 - 1)(5 \cdot 4 + 3)}}{2}$$

$$= \frac{2.65331193\ldots}{2}$$

$$< 3.$$

So, we have the following result:
Proposition 2.1 There are no natural numbers \( y \) and \( z \), and \( n \geq 3 \), such that:

\[
4^n + y^n = z^n.
\] (2.8)

We also observe that, as \( x \to \infty \), the right-hand side of (2.7) grows like:

\[
\frac{-x + \sqrt{x \cdot 5x}}{2} = \frac{-1 + \sqrt{5}}{2} \cdot x = 0.618033988 \cdots x = (\varphi - 1)x = \frac{1}{\varphi}x,
\]

where \( \varphi \) represents the Golden ratio. Thus, for example, for \( x = 100 \), constraint (2.7) becomes:

\[
n \leq \frac{1 - 100 + \sqrt{(100 - 1)(5 \cdot 100 + 3)}}{2} = 62.0762071 \cdots
\]

Therefore, we obtain:

\[
n \leq 62.
\] (2.9)

Let us find now a constraint for \( y \). From the inequality:

\[
x^n \geq ny^{n-1}k,
\] (2.10)

we obtain:

\[
y \leq x \cdot \left(\frac{x}{n}\right)^{1/(n-1)}.
\] (2.11)

Finally, let us find a constraint for \( k \). From the inequality:

\[
x^n \geq ny^{n-1}k,
\]

we obtain:

\[
k \leq \left(\frac{x}{y}\right)^{n-1} \cdot \frac{x}{n}.
\] (2.12)
Finally, since the numbers $x^n$, $y^n$, and $z^n$, are too big to be computed, by dividing equation (1.1) by $y^n$, we obtain:

$$\left(\frac{x}{y}\right)^n + 1 = \left(1 + \frac{k}{y}\right)^n. \quad (2.13)$$

Observe that, from the conditions that we imposed:

$$1 > \left(\frac{x}{y}\right)^n = \frac{x^n}{y^{n-1}} \cdot \frac{1}{y} \geq \frac{nk}{y},$$

if $y$ is not gigantic, then $nk/y$ can be detected by the computer as a non–zero number.

Also, since $nk < x < y$, we have:

$$1 < \left(1 + \frac{k}{y}\right)^n = \left(1 + \frac{nk}{ny}\right)^n < \left(1 + \frac{y}{ny}\right)^n = \left(1 + \frac{1}{n}\right)^n < e.$$

Since $[1 + (k/y)]^n$ does not grow indefinitely, the computer can calculate it.

With these new things in place, the proposed algorithm is:

for $x = 5$ to $x_{\text{max}}$ (for reasons that will be explained later $x_{\text{max}} = 50$),

for $n = 3$ to $\left\lfloor \frac{1 - x + \sqrt{(x - 1)(5x + 3)}}{2} \right\rfloor$,

for $y = x$ to $\left\lfloor x \left(\frac{x}{n}\right)^{1/(n-1)} \right\rfloor$,

for $k = 1$ to $\left\lfloor \left(\frac{x}{y}\right)^{n-1} \frac{x}{n} \right\rfloor$, 


check whether:
\[
\left(\frac{x}{y}\right)^n + 1 = \left(1 + \frac{k}{y}\right)^n
\]
and record:
\[
\gamma := -\frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x}.
\]

3 An elementary inequality

In this section we prove the following inequality.

{} • If \(0 < r < 1\), then for all \(t \in [0, 1]\), we have:

\[
1 + (2^r - 1) t \leq (1 + t)^r \leq 1 + (2^r - 1) t^{2r-1/(2r-1)}.
\]

Moreover, \(\alpha := 1\) is the least among all \(\alpha\), and \(\beta := r \cdot 2^{r-1}/(2r - 1)\) is the greatest among all \(\beta\), such that the inequality:

\[
1 + (2^r - 1) t^\alpha \leq (1 + t)^r \leq 1 + (2^r - 1) t^\beta
\]
holds for all \(t \in [0, 1]\).

• If \(r > 1\), then for all \(t \in [0, 1]\), we have:

\[
1 + (2^r - 1) t^{r-2^{r-1}/(2r-1)} \leq (1 + t)^r \leq 1 + (2^r - 1) t.
\]

Moreover, \(\alpha := r \cdot 2^{r-1}/(2r - 1)\) is the least among all \(\alpha\), and \(\beta := 1\) is the greatest among all \(\beta\), such that the inequality:

\[
1 + (2^r - 1) t^\alpha \leq (1 + t)^r \leq 1 + (2^r - 1) t^\beta
\]
holds for all \(t \in [0, 1]\).

**Proof.** Let us assume that \(0 < r < 1\). To prove the inequality:

\[
1 + (2^r - 1) t \leq (1 + t)^r,
\]
for all \(t \in [0, 1]\), let us observe that the function \(f : [0, 1] \to \mathbb{R}\), \(f(t) = (1+t)^r\) is concave downward (since \(0 < r < 1\), while the function \(g : [0, 1] \to \mathbb{R}\),
\( g(t) = 1 + (2^r - 1)t \) is linear. Since \( g(0) = f(0) = 1 \) and \( g(1) = f(1) = 2^r \), and \( f \) is concave downward, the graph of \( f \) lies above the line segment joining the points \((0, f(0))\) and \((1, f(1))\). Because \( g \) is linear, this line segment is the graph of \( g \). Thus we have:

\[
g(t) \leq f(t),
\]

for all \( t \in [0, 1] \).

To prove the inequality:

\[
(1 + t)^r \leq 1 + (2^r - 1) t^{r-2^r-1/(2^r-1)},
\]

for all \( t \in [0, 1] \), let us consider the function \( h : [0, 1] \to \mathbb{R} \), defined as:

\[
h(t) = (1 + t)^r - 1 - (2^r - 1) t^{r-2^r-1/(2^r-1)}.
\]

We have \( h(0) = h(1) = 0 \). Its derivative is:

\[
h'(t) = r \left[ (1 + t)^{r-1} - 2^{r-1} t^{r-2^r-1/(2^r-1)-1} \right],
\]

for all \( t \in (0, 1] \).

Let us observe that the exponent of \( t \) in \( h'(t) - r(1 + t)^{r-1} \) is:

\[
\frac{r \cdot 2^{r-1}}{2^r - 1} - 1 = \frac{1 - (2 - r)2^{r-1}}{2^r - 1} = \frac{2^{1-r} - (2 - r)}{2 - 2^{1-r}} = \frac{(1 + 1)^{1-r} - [1 + (1 - r)]}{2 - 2^r} < 0,
\]

due to Bernoulli inequality:

\[
(1 + x)^{1-r} < 1 + (1 - r)x,
\]

applied for \( x = 1 \). Thus, we have:

\[
\lim_{t \to 0^+} h'(t) = r \left[ \lim_{t \to 0^+} (1 + t)^{r-1} - 2^{r-1} \lim_{t \to 0^+} t^{r-2^r-1/(2^r-1)-1} \right] = -\infty.
\]
Let us also observe that:

\[ h'(1) = 0. \]  

(3.10)

We observe that \( h'(t) > 0 \) is equivalent to:

\[ (1 + t)^{r-1} > 2^{r-1}t^{-2r-1}/(2^r-1)^{-1}, \]  

(3.11)

which after raising both sides of this inequality to the negative power \( 1/(r - 1) \), and taken \( \ln \) from both sides becomes:

\[ \ln(1 + t) < \ln 2 + \frac{2^{1-r} - (2 - r)}{(2 - 2^{1-r})(r - 1)} \ln t. \]  

(3.12)

This is equivalent to:

\[ \frac{L(t, 1)}{L(1 + t, 2)} > \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r})(1 - r)}, \]  

(3.13)

where \( L(a, b) \) denotes the logarithmic mean of \( a \) and \( b \), defined as:

\[ L(a, b) := \frac{b - a}{\ln b - \ln a}, \]  

(3.14)

for any \( a \) and \( b \) positive numbers, such that \( a \neq b \), and

\[ L(a, a) := a, \]  

(3.15)

for all \( a > 0 \).

Let us consider the function \( F : (0, 1] \rightarrow \mathbb{R} \), defined by:

\[ F(t) := \frac{L(t, 1)}{L(1 + t, 2)}. \]  

(3.16)

**Claim 1:** The function \( F \) is increasing on \((0, 1]\).

One way, to see this, is to compute the derivative of \( F \) as:

\[
F'(t) = \left[ \frac{\ln 2 - \ln(1 + t)}{-\ln t} \right]' \\
= \frac{1}{\ln^2 t} \cdot \left[ \frac{\ln t + \ln 2 - \ln(1 + t)}{1 + t} - \frac{t}{t} \right] \\
= \frac{1}{\ln^2 t} \cdot \frac{(1 + t) \ln 2 - [(1 + t) \ln(1 + t) - t \ln(t)]}{t(1 + t)} \\
> 0,
\]
for all $t \in (0, 1)$, for the following reason. The function $G(t) := (1 + t) \ln 2 - [(1 + t) \ln(1 + t) - t \ln t]$ is concave upward on $[0, 1]$, its second derivative being $G''(t) = 1/[t(t + 1)] > 0$. Thus $G'$ is increasing on $[0, 1]$, which means, for all $t \in [0, 1]$, we have $G'(t) \leq G'(1) = 0$. Hence $G$ is decreasing on $[0, 1]$, which means, for all $t \in [0, 1]$, $G(t) \geq G(1) = 0.$

Another way is to observe that, for all $0 < t_1 < t_2 < 1$, proving that:

$$\frac{L(t_1, 1)}{L(1 + t_1, 2)} < \frac{L(t_2, 1)}{L(1 + t_2, 2)}$$

is equivalent to:

$$\frac{\int_{t_1}^{1} \frac{1}{(s + 1)ds}}{\int_{t_1}^{1} \frac{1}{sds}} < \frac{\int_{t_2}^{1} \frac{1}{(s + 1)ds}}{\int_{t_2}^{1} \frac{1}{sds}}.$$

The last inequality is equivalent after doing the cross multiplication and writing:

$$\int_{t_1}^{1} \frac{1}{s + r} ds = \int_{t_1}^{t_2} \frac{1}{s + r} ds + \int_{t_2}^{1} \frac{1}{s + r} ds, \quad (3.17)$$

for $r \in \{0, 1\}$, to:

$$\int_{t_1}^{t_2} \frac{1}{s + 1} ds \int_{t_2}^{1} \frac{1}{s} ds < \int_{t_1}^{t_2} \frac{1}{s} ds \int_{t_2}^{1} \frac{1}{s + 1} ds.$$

Finally, the last inequality can be written as the following double integral inequality:

$$\int_{t_1}^{t_2} \int_{t_2}^{1} \frac{1}{x + 1} \cdot \frac{1}{y} dy dx < \int_{t_1}^{t_2} \int_{t_2}^{1} \frac{1}{x} \cdot \frac{1}{y + 1} dy dx.$$

Moving all terms to the left and using the linearity of the double integral, the last inequality is equivalent to:

$$\int_{t_1}^{t_2} \int_{t_2}^{1} \frac{x - y}{x(x + 1)y(y + 1)} dy dx < 0,$$

which is clearly true since $x \leq t_2 \leq y$.

Claim 2: We have:

$$\lim_{t \to 0^+} F(t) < \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r})(1 - r)}. \quad (3.18)$$
Indeed, using Bernoulli inequality, we have:

\[
\lim_{t \to 0^+} F(t) = \lim_{t \to 0^+} \frac{\ln 2 - \ln(1 + t)}{\ln 1 - \ln t} = 0 \]

\[
< \frac{[1 + (1 - r)1] - (1 + 1)^1 - r}{(2 - 2^{1-r}) (1 - r)} = \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r}) (1 - r)}.
\]

**Claim 3:** We have:

\[
\lim_{t \to 1^-} F(t) > \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r}) (1 - r)}. \tag{3.19}
\]

Indeed, we have:

\[
\lim_{t \to 1^-} F(t)
= \lim_{t \to 1^-} \frac{\ln (2/(1 + t))}{\ln(1/t)}
= \lim_{t \to 1^-} \frac{\ln (1 + (1 - t)/(1 + t))}{\ln (1 + (1 - t)/t)}
= \lim_{t \to 1^-} \frac{\ln (1 + (1 - t)/(1 + t))}{(1 - t)/(1 + t)} \cdot \lim_{t \to 1^-} \frac{(1 - t)/t}{\ln (1 + (1 - t)/t)} \cdot \lim_{t \to 1^-} \frac{t}{1 + t}
= 1 \cdot 1 \cdot \frac{1}{2}
= \frac{1}{2},
\]

since we have the fundamental limit:

\[
\lim_{s \to 0} \frac{\ln(1 + s)}{s} = 1.
\]

To prove our last claim, we need to check that:

\[
\frac{1}{2} > \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r}) (1 - r)}. \tag{3.20}
\]

The last inequality is equivalent to:

\[
(1 - 2^{-r}) (1 - r) > 2 - r - 2^{1-r}. \tag{3.21}
\]
which, after performing the multiplication in the left, reduces to proving that:

\[-2^{-r} + r \cdot 2^{-r} > 1 - 2^{1-r},\]  
(3.22)

or equivalently, after multiplying both sides by $2^r$:

\[1 + r > 2^r.\]  
(3.23)

The last inequality follows from Bernoulli inequality:

\[1 + rx > (1 + x)^r,\]  
(3.24)

for all $x > 0$, applied for $x := 1$.

Since $F$ is continuous on $(0, 1)$, by the Intermediate Value Theorem, it follows from the last three claims that, there exists a unique point $t_0 \in (0, 1)$, such that:

\[F(t_0) = \frac{(2 - r) - 2^{1-r}}{(2 - 2^{1-r})(1 - r)},\]

and for all $t \in (0, t_0)$, we have $F(t) < F(t_0)$, while for all $t \in (t_0, 1)$, we have $F(t) > F(t_0)$.

Therefore, the function $h(t) = (1 + t)^r - 1 - (2^r - 1)t^{r-2^r-1/(2^r-1)}$ is decreasing on $[0, t_0]$, and increasing on $[t_0, 1]$. Thus, its maximum value is $\max\{h(0), h(1)\} = 0$, and so we have that, for all $t \in [0, 1]$:

\[(1 + t)^r \leq 1 + (2^r - 1)t^{r-2^r-1/(2^r-1)}.\]  
(3.25)

Let us prove now the optimality of the powers $\alpha = 1$ and $\beta = r \cdot 2^{r-1}/(2^r - 1)$. Let us suppose that there exists $\alpha < 1$, such that, for all $t \in (0, 1)$, we have:

\[1 + (2^r - 1)t^\alpha \leq (1 + t)^r.\]

Then subtracting first 1 from both sides of this inequality, then dividing both sides by $t$, and passing to the limit as $t \to 0^+$, we obtain:

\[(2^r - 1) \lim_{t \to 0^+} t^{\alpha-1} \leq \lim_{t \to 0^+} \frac{(1 + t)^r - 1}{t}.\]

The last inequality is equivalent to:

\[\infty \leq r, 1\]
which is a contradiction.

Let \( \beta > 0 \) be such that, for all \( t \in (0, 1) \), we have:

\[
(1 + t)^r \leq 1 + (2^r - 1) t^\beta.
\]

Then subtracting first \( 2^r \) from both sides of this inequality, then dividing both sides by the negative number \( t - 1 \), and passing to the limit as \( t \to 1^- \), we obtain:

\[
\lim_{t \to 1^-} \frac{(1 + t)^r - 2^r}{t - 1} \geq (2^r - 1) \lim_{t \to 1^-} \frac{t^\beta - 1}{t - 1}.
\]

Observing that the above limits are derivatives evaluated at 1, we conclude that:

\[
r \cdot 2^{r-1} \geq (2^r - 1) \beta.
\]

That means:

\[
\beta \leq \frac{r \cdot 2^{r-1}}{2^r - 1}.
\]

The part of this Proposition, corresponding to \( r > 1 \), can be proven similarly, observing that the inequalities used in the first part are reversed. \( \Box \)

**Observation 3.2** Since, according to Bernoulli inequality, for \( 0 < r < 1 \), we have:

\[
2^r - 1 < r,
\]

the inequality:

\[
(1 + t)^r \leq 1 + (2^r - 1) t^{r-2^r/(2^r-1)}
\]

is weaker than Bernoulli inequality:

\[
(1 + t)^r \leq 1 + rt,
\]

for \( t \) in \([0, \epsilon)\), and stronger than Bernoulli inequality, for \( t \) in \((1 - \epsilon, 1]\), where:

\[
\epsilon := \left( \frac{2^r - 1}{r} \right)^{(2^r-1)/(2^r-r2^{r-1}-1)} \in (0, 1).
\]
4 A refined algorithm

In this section, for a fixed \( x \geq 50 \) (assuming that we have checked Fermat’s Theorem for all \( x \leq 49 \) using the first algorithm), we try to find better margins for \( n, y, \) and \( k \).

We can write Fermat’s equation (1.1) as:

\[
 x^n = \int_y^z nt^{n-1} dt, \tag{4.1}
\]

and dividing both sides by \( n \), we obtain:

\[
 \frac{x^n}{n} = \int_y^z t^{n-1} dt. \tag{4.2}
\]

Since \( n \geq 2 \), the function \( f(t) = t^{n-1} \) is concave upward on \([y, z]\). Therefore, applying Hermite–Hadamard inequality, we have:

\[
 \int_y^z t^{n-1} dt \geq (z - y) f \left( \frac{y + z}{2} \right), \tag{4.3}
\]

which is equivalent to:

\[
 \int_y^z t^{n-1} dt \geq k \left( y + \frac{k}{2} \right)^{n-1}. \tag{4.4}
\]

Thus, we obtain the inequality:

\[
 \frac{x^n}{nk} \geq \left( y + \frac{k}{2} \right)^{n-1}. \tag{4.5}
\]

Multiplying both sides by \( nk/x^{n-1} \), and taking \( \ln \) from both sides of the resulting inequality, we obtain:

\[
 \ln x \geq \ln n + \ln k + (n - 1) \ln \left( \frac{y}{x} + \frac{k}{2x} \right). \tag{4.6}
\]

Since \( y \geq x + 1 \) and \( k \geq 1 \), \( \ln k \geq 0 \), and it follows from the last inequality that:

\[
 \ln x \geq \ln n + (n - 1) \ln \left( 1 + \frac{3}{2x} \right). \tag{4.7}
\]
The last inequality can be rewritten in terms of the logarithmic mean as:

\[
\frac{1}{x-n} \cdot \frac{x-n}{\ln x - \ln n} \leq \frac{2x}{3} \cdot \frac{1}{n-1} \cdot \frac{1 + 3/(2x) - 1}{\ln (1 + 3/(2x)) - \ln 1}.
\] (4.8)

That means:

\[
\frac{1}{x-n} \cdot L(n, x) \leq \frac{2x}{3} \cdot \frac{1}{n-1} \cdot L\left(1, 1 + \frac{3}{2x}\right).
\] (4.9)

Using the known fact that the logarithmic mean of two positive numbers is greater than their geometric mean, and less than their arithmetic mean, we obtain:

\[
\frac{1}{x-n} \cdot \sqrt{nx} \leq \frac{2x}{3} \cdot \frac{1}{n-1} \cdot \left(1 + \frac{3}{4x}\right).
\] (4.10)

That means:

\[
\frac{\sqrt{nx}}{x-n} \leq \frac{4x + 3}{6(n-1)}.
\] (4.11)

Let us denote:

\[
\alpha := \frac{n}{x}.
\] (4.12)

Then the last inequality becomes:

\[
\frac{\sqrt{\alpha}}{1-\alpha} \leq \frac{4x + 3}{6\alpha x - 6}.
\] (4.13)

This is equivalent to:

\[
(6\alpha \sqrt{\alpha} + 4\alpha - 4)x \leq 6 - 3(\sqrt{\alpha} - 1)^2.
\] (4.14)

Since \((\sqrt{\alpha} - 1)^2\) is positive, we obtain:

\[
(6\alpha \sqrt{\alpha} + 4\alpha - 4)x \leq 6.
\] (4.15)

That means:

\[
(3\alpha \sqrt{\alpha} + 2\alpha - 2)x \leq 3.
\] (4.16)
If we assume that $\alpha \geq 1/2$, then:

$$
\left( \frac{3}{2\sqrt{2}} + 2 \cdot \frac{1}{2} - 2 \right) x \leq 3,
$$

which implies:

$$
x \leq \frac{3}{3/(2\sqrt{2}) - 1}.
$$

That means $x \leq 49.4558\ldots$, which is not possible since we have assumed that $x \geq 50$.

Therefore, we conclude that for $x \geq 50$, the ratio $n/x$ must be smaller than 1/2, and so we have the following margins for $n$:

$$3 \leq n \leq \lfloor x/2 \rfloor.\quad (4.19)$$

We will find now a margin for $k$. We proceed as follows:

$$
\left(1 + \frac{k}{y}\right)^n = 1 + \left(\frac{x}{y}\right)^n.\quad (4.20)
$$

Raising both sides to the power $1/n$ and using the inequality that we proved in section 3, for $r = 1/n \in (0, 1)$ and $t = (x/y)^n \in (0, 1)$, we obtain:

$$
1 + \frac{k}{y} = \left[ 1 + \left(\frac{x}{y}\right)^n \right]^{1/n} \\
\leq 1 + \left(2^{1/n} - 1\right)\left(\frac{x}{y}\right)^{n(1/n)\cdot 2^{(1/n)-1}/(2^{(1/n)-1})}.\quad (4.21)
$$

From here we conclude that:

$$k \leq \left(2^{1/n} - 1\right)\left(\frac{x}{y}\right)^{2^{(1/n)-1}/(2^{(1/n)-1})} \cdot y.\quad (4.22)$$

That means:

$$k \leq \left(2^{1/n} - 1\right)\left(\frac{x}{y}\right)^{(1-2^{(1/n)-1})/(2^{(1/n)-1})} \cdot x.\quad (4.22)$$
Since $y \geq x + 1$, the last inequality implies:

$$k \leq (2^{1/n} - 1) \left( \frac{x}{x+1} \right)^{(1-2(1/n)^{-1})/(2^{1/n} - 1)} \cdot x. \quad (4.23)$$

In particular, since:

$$\frac{1 - 2(1/n)^{-1}}{2^{1/n} - 1} > 0, \quad (4.24)$$

we obtain:

$$k < (2^{1/n} - 1) x \quad (4.25)$$

$$< \frac{1}{n} x. \quad (4.26)$$

We will find now a lower and an upper bound for $y$. Let $\mu$ be the probability measure given by the density function

$$\varphi(t) := \frac{1}{k} 1_{[y,z]}(t), \quad (4.27)$$

where $1_{[y,z]}$ denotes the characteristic function of the interval $[y, z]$. Using the Lyapunov inequality, which says that for a probability measure $\mu$, the $L^p$-norm, with respect to $\mu$, of a random variable is an increasing function of $p$, since $n - 1 \geq 2$, we have:

$$\left( \frac{x^n}{kn} \right)^{1/(n-1)} = \left( \frac{1}{k} \int_y^{y+k} t^{n-1} dt \right)^{1/(n-1)} \geq \left( \frac{1}{k} \int_y^{y+k} t^2 dt \right)^{1/2} = \sqrt{y^2 + ky + \frac{k^2}{3}}. \quad (4.28)$$

We would like to make the observation that since:

$$\sqrt{y^2 + ky + \frac{k^2}{3}} \geq y + \frac{k}{2}, \quad (4.29)$$
inequality (4.28) is a little bit better than (4.5).
It follows from (4.28) that $y$ has to be less than or equal to the positive root of the quadratic function:

$$
q(s) := s^2 + ks + \frac{k^2}{3} - \left(\frac{x^n}{kn}\right)^{2/(n-1)}.
$$

(4.30)

That means:

$$
y \leq -k + \sqrt{4[x^n/(kn)]^{2/(n-1)} - (k^2/3)} \quad (4.31)
$$

On the other hand, we have:

$$
\frac{x^n}{nk} = \frac{1}{k} \int_y^{y+k} t^{n-1} dt
\leq \frac{1}{k} \int_y^{y+k} (y+k)^{n-1} dt
= (y+k)^{n-1},
$$

(4.32)

We conclude that:

$$
y > x \left(\frac{x}{nk}\right)^{1/(n-1)} - k. \quad (4.33)
$$

Using inequality (3.1), we have:

$$
1 + \frac{k}{y} = \left(1 + \frac{x^n}{y^n}\right)^{1/n} \geq 1 + (2^{1/n} - 1) \frac{x^n}{y^n}.
$$

(4.34)

From here we obtain:

$$
y \geq \left[\frac{(2^{1/n} - 1)x}{k}\right]^{1/(n-1)} x
> x,
$$

(4.35)

due to (4.25). Therefore, using (4.33) and (4.35), we conclude that:

$$
y \geq \max \left\{\left(\frac{x}{nk}\right)^{1/(n-1)} - \frac{k}{x} \left[\frac{(2^{1/n} - 1)x}{k}\right]^{1/(n-1)}\right\} \cdot x. \quad (4.36)
$$
Therefore, the new proposed algorithm, for \( x \geq 50 \), is:

for \( n = 3 \) to \( n = \lfloor \frac{x}{2} \rfloor \),

for \( k = 1 \)

to \( k = \lfloor (2^{1/n} - 1) \left( \frac{x}{x + 1} \right)^{(1-2(1/n)-1)/(2(1/n)-1)} \cdot x \rfloor \),

for \( y = \lceil \max \left\{ \left( \frac{x}{nk} \right)^{1/(n-1)} - \frac{k}{x}, \left[ \frac{(2^{1/n} - 1)x}{k} \right]^{1/(n-1)} \right\} \cdot x \rceil \)

to \( \lfloor -k + \sqrt{4[x^n/(kn)]^{2/(n-1)} - (k^2/3)} \rfloor / 2 \),

check whether:

\[
\left( \frac{x}{y} \right)^n + 1 = \left( 1 + \frac{k}{y} \right)^n
\]

and record:

\[
\gamma := -\frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x}.
\]

5 Complexity Considerations

We would like to discuss a little bit about the complexity of the second algorithm, which works for \( x \geq 50 \).

If we neglect the number of additions, subtractions, multiplications, divisions, radical extractions, and rasing to power operations, and count only the number of cycles, then for a fixed \( x \), we can say the following about our algorithm:

Since \( 3 \leq n \leq x/2 \), \( 1 \leq k < (2^{1/n} - 1)x \), and \( A - k < y \leq A - (k/2) \), where:

\[
A := \left( \frac{x}{nk} \right)^{1/(n-1)} \cdot x, \quad (5.1)
\]

and because of the following limits:

\[
\lim_{n \to \infty} \frac{2^{1/n} - 1}{1/n} = \ln 2 \quad (5.2)
\]

and

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{j} - \ln n \right) = c, \quad (5.3)
\]
where \( c \) denotes Euler’s constant, the number of cycles in our algorithm grows at most like:

\[
\sum_{n=3}^{\left\lfloor x/2 \right\rfloor} \sum_{k=1}^{\lfloor (2^{1/n}-1)x \rfloor} \frac{k}{2} \\
\approx \frac{1}{2} \sum_{n=3}^{\left\lfloor x/2 \right\rfloor} \frac{\left(2^{1/n}-1\right)x \left[\left(2^{1/n}-1\right)x + 1\right]}{2} \\
\approx \frac{x}{4} \sum_{n=3}^{\left\lfloor x/2 \right\rfloor} \ln 2 \left(\frac{\ln 2}{n} x + 1\right) \\
\approx \frac{1}{4} \left[ x^2 \left(\ln 2\right)^2 \sum_{n=3}^{\left\lfloor x/2 \right\rfloor} \frac{1}{n^2} + x \ln 2 \sum_{n=3}^{\left\lfloor x/2 \right\rfloor} \frac{1}{n} \right] \\
\approx \frac{1}{4} \left[ x^2 \left(\ln 2\right)^2 \left(\frac{\pi^2}{6} - 1 - \frac{1}{2^2}\right) + x \ln 2 \left(\ln \left(\frac{x}{2}\right) + c - 1 - \frac{1}{2}\right) \right].
\]

So, we can say that our algorithm has an order of \( O(x^2) \).

We would like to mention that there are sharper inequalities than the one relating the logarithmic mean and the arithmetic mean of two positive numbers, in the sense that the arithmetic mean can be replaced by smaller Hölder means, see [3], [5], and [4], but we believe that those inequalities are a little bit impractical to use, and their use do not speed up too much our algorithm. We are including below the maximum value for \( \gamma \) computed by both first and second algorithm. In what follows

\[
\gamma := \sup \left\{ \frac{\ln |1 + (x/y)^n - (z/y)^n|}{\ln x} \mid 5 \leq x \leq x_{MAX} \right\}, \quad (5.4)
\]

where \( x_{MAX} \) is the maximum value assigned to \( x \).
6 Numerical Results

Table 1: First Algorithm

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<th>xmin</th>
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<td>500</td>
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<td>230589149</td>
<td>1895272741</td>
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<td>4.435969439</td>
<td>4.435969439</td>
<td>4.435969439</td>
<td>4.435969439</td>
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</table>

Table 2: Second Algorithm

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<tbody>
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<tr>
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<td>3.475021756</td>
<td>3.725872227</td>
<td>3.725872227</td>
<td>3.725872227</td>
</tr>
</tbody>
</table>

Important observation: the highest value of γ (4.435969439) is attained in the lower range of xmax. The algorithm was actually ran with xmax values beyond 3000 and γ never increased.

Note: From x = 50 to x = 3000 it takes 12 – 36 hours, depending on the machine speed.
7 Conjecture

Finally, we would like to end the paper with a conjecture, about the exponent $\gamma$ suggested by the numerical algorithm from this paper, namely:

**Conjecture 7.1** Under the bounds, obtained in this paper, for $n \in [3, N(x)] \cap \mathbb{N}$, $y \in [x+1, Y(x+1)] \cap \mathbb{N}$, and $k \in [1, K(x)] \cap \mathbb{N}$, relative to $x$, there exists a finite positive number $\gamma \approx 4.435969439$, such that:

$$\inf \left\{ \frac{\ln |1 + \left(\frac{x}{y}\right)^n - \left(\frac{z}{y}\right)^n|}{\ln x} \right\} = -\gamma. \quad (7.1)$$

We hope that this conjecture, which may be even harder to prove than Fermat’s Last Theorem, will make this celebrated theorem interesting for people working in the area of Analysis. Moreover, as this paper humbly shows, Fermat’s Last Theorem, even though proven by Andrew Wiles, may still continue to stimulate new mathematical research.

References


