3D Graphical Representation of Elementary Real Variable Functions with Extended Domain

Gonzalo Urcid*[1] and Gustavo U. Bautista[2]

[1]Optics Department, INAOE, Tonanzintla 72840, Pue., Mexico
[2]Chemical Engineering Faculty, UP, Puebla 72570, Pue., Mexico

*Corresponding Author:
Email: gurcid@inaoep.mx

Abstract
In this work, we introduce an alternative graphical representation of elementary functions of a real variable whose natural domain can be extended to include almost all real numbers, except those numbers where the function is discontinuous. The usual graphical representation of such numerical functions, whose domain is a proper subset of $\mathbb{R}$ *shows one or several curvilinear segments drawn in the two-dimensional* xy *plane. However, if the natural domain of a given function is extended to* $\mathbb{R}$ *or almost all* $\mathbb{R}$*, its evaluation may result in a complex number for those real numbers belonging to its domain extension. A process, equivalent in a natural way, to represent graphically a complex function of a real variable. In this case, we give a novel graphic representation in three-dimensional* xyz *space, in which real and complex numbers coexist and can be displayed simultaneously. A simple theoretical explanation is provided to support this idea. We include various examples of elementary functions whose natural domain and range has been extended with the purpose of illustrating the type of spatial graphs proposed.*

Keywords: domain extended functions, elementary functions, function graphs, graphs of complex functions of a real variable, spatial graphs of elementary functions

1. Introduction

Supported by its graphical representations, the mathematical concept of function is considered as one of the key ideas in general mathematics and its usefulness in several of its pure and applied branches is beyond doubt. This concept is commonly explained across the different study programs from high school to the first two years of university education in science and engineering programs. In high school, the function concept together with its graphical representation in the *xy* plane can be introduced, within the context of basic algebra, in relation to the solution of first and second degree algebraic equations (straight line and parabola). It can be further developed as part of trigonometry in relation to the circular functions (sine, cosine, tangent) as well as in analytic plane geometry through the study of elementary geometrical loci (circumference, ellipse, hyperbola). Also, a more detailed treatment about elementary functions and their applications to diverse practical problems can be realized at this educational stage. At the undergraduate level, numerical functions and their two dimensional graphic representation can be treated to serve as a tool in basic algebra, pre-calculus, or as background material for an in-depth study of differential and integral calculus [1],[2]. A general brief exposition about real functions of one or several variables is given in [3] and the same topic, considered from an advanced didactic point of view, can be consulted in [4].

A related topic is the graphical representation of complex functions whose study is reserved for courses in higher or advanced mathematics. Thus, within the realm of the complex variable calculus, complex transformations and conformal mapping are specific subjects that extend the well known graphical representation of real functions of one variable and is commonly supported by auxiliary visualization techniques [5]-[9]. Today, almost all of the different kinds of graphical representations of real and complex functions in two or three dimensional space, some including animated effects, are already available in multitude of software applications and visualization tools such as those listed in [10]-[18].

The present work is organized as follows: Section 2 gives a brief mathematical background about functions with the purpose of establishing the conceptual framework and the appropriate symbolism for the ideas to be developed in Section 3, that explains in detail, the concept of an extended elementary function. In Section 4 we describe a new graphical representation in three-dimensional space of the extended elementary functions previously discussed. Finally, the conclusions about the material here exposed are commented in Section 5.

2. Elementary Functions

Recall that a mathematical *function* is a triplet of objects (A, B, f), of which f is a rule that assigns to each element x of set A a single element y of set B. Therefore, we write $y = f(x)$, meaning that y is the value or the image of x under f. The departure set A is named the *domain* of f and the arrival set B is known as the *codomain* of f. The assignment rule f determines set A over which the function is defined and consequently we can abbreviate (A, B, f) by f. In case that sets A and B should be made explicit, the arrow notation, $f: A \longrightarrow B$ is used to speak of function f with variable in A taking values in B. The collection of all y values obtained by applying f to each $x \in A$ is the *range* or *image* of A under f, and satisfies the set inclusion relation, $f(A) \subseteq B$. In this paper, the specific case of interest considers that $A, B \subseteq \mathbb{R}$ (subsets of the set of real numbers), i. e., *real functions of a real variable*, such as, the algebraic or transcendental functions. By an *elementary function* of the aforementioned types we mean a function whose mathematical expression in x results from the composition of a finite number of arithmetical operations with constants, exponentials, logarithmic or the extraction of roots (radicals). Specifically, the functions used in this work are listed in Table 1 and we assume that the reader has a basic knowledge about their characteristics as well as the Cartesian graphical representation in the xy plane (see, e. g. [1]-[4]). In Table 1, c denotes a real constant. Also, notice that, except for the cubic function, for the other listed functions the domain A and the codomain B are not necessarily both equal to $\mathbb{R}$.

Table 1. Description of some elementary real functions of a real variable;
$\mu = 1/e = \min\{x^x : x \in A\}$.

Name	**Rule** $f(x)$	**Domain** A	**Range** $f(A)$	**Type**
Quadratic (parabola)	$x^2 + c$	$\mathbb{R}$	$[c, \infty)$	2nd degree polynomial
Cubic	$x^3 + c$	$\mathbb{R}$	$\mathbb{R}$	3rd degree polynomial
Square root	$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$	simple radical
Semicircle	$\sqrt{1 - x^2}$	$[-1,1]$	$[0,1]$	compound radical
Inverse of semicircle	$1/\sqrt{1 - x^2}$	$(-1,1)$	$[1, \infty)$	rational radical
Natural logarithm	$\ln x$	$(0, \infty)$	$\mathbb{R}$	logarithm base e
Spindle	x^x	$[0, \infty)$	$[\mu, \infty)$	power exponential

Therefore, based on the assignment rule $f(x)$, we see that $A \subseteq \mathbb{R}$ and $f(A) \subseteq B = \mathbb{R}$. Also, if $A = \mathbb{R}$, the range of f is determined by the values $f(x)$ obtained for all $x \in A$ and forms a subset of $\mathbb{R}$. However, for the last four functions listed in Table 1, if $x \in \mathbb{R}$ but $x \notin A$, f can still be evaluated for almost any x giving as result a complex number, which in turn allows the possibility of extending the natural domain A to $\mathbb{R}$, except for those values of x that coincide with non-removable discontinuities (vertical asymptotes).

3. Extended Elementary Functions

By the comments made in the last paragraph of Section 2, it is not difficult to define the notion of an extended elementary function by considering a "double" extension. First the domain A of f is extended by aggregating those numbers $x \in \mathbb{R}$ such that $y = f(x)$ is a complex number, i.e., such that $y \in \mathbb{C}$. If $\tilde{A}$ denotes the set of all such elements of $\mathbb{R}$, then the new domain of f is given by $A \cup \tilde{A}$ and may cover all real numbers except for a set of discontinuities with null measure. Naturally, the second extension results from enlarging codomain B to the set $\mathbb{C}$ of complex numbers by considering the range values y that are obtained by applying function f to $\tilde{A}$. Thus, an *extended elementary function* is the new object triplet $(\tilde{f},\ A \cup \tilde{A}, \tilde{B})$, where $A \cup \tilde{A}$ stands for the extended domain, $\tilde{B} = \mathbb{C}$ is the enlarged codomain, and the assignment rule $\tilde{f}$ is expressed separately in terms of the original elementary function f, respectively, over A and $\tilde{A}$. The extended versions of the last four elementary functions listed in Table 1 are given in Table 2, where $i = \sqrt{-1}$ denotes the imaginary unit. It is clear that $\tilde{f}(x) = f(x)$ if $x \in A$ and that $\tilde{f}(x)$ is an imaginary or complex number whenever $x \in \tilde{A}$. Because of its numerical nature, note that the range of the extended domain appears as two separate sets, $f(A)$ and $f(\tilde{A})$ whose union equals $\tilde{f}(A \cup \tilde{A})$.

A non-trivial elementary real function, whose domain and range are both not equal to $\mathbb{R}$, is the spindle function, also known as the 2nd tetration [19] or the power tower of order 2 function [20]. Its domain and range as given in Table 1 are the sets commonly used in elementary courses. However, further analysis in the ways the spindle function is evaluated numerically shows that is natural domain and true range are indeed more complicated to describe as subsets of $\mathbb{R}$. Specifically, in relation to Table 2 and following [21], we have that $x \in A_n$, if x is a negative rational number $-p/q$, where p and q are positive integers and q is odd. Also, recall that $A^c = \mathbb{R} - A$ (complement of set A w.r.t. $\mathbb{R}$), and that $\lim_{x \to 0} x^x = 1$. For a complete discussion of the spindle function peruse [21].

Table 2. Description of some extended elementary functions of a real variable (see text for A_n and A^c).

Name	**Rule** $\tilde{f}(x)$	**Domain** $A \cup \tilde{A}$	**Splitted range**	**Type**
Square root	$\sqrt{x}$	$A = [0, \infty)$	$f(A) = [0, \infty)$	real
	$i\sqrt{-x}$	$\tilde{A} = (-\infty, 0)$	$f(\tilde{A}) = (0, \infty)$	imaginary
Semicircle	$\sqrt{1 - x^2}$	$A = [-1,1]$	$f(A) = [0,1]$	real
	$i\sqrt{x^2 - 1}$	$\tilde{A} = (-\infty, -1) \cup (1, \infty)$	$f(\tilde{A}) = (0, \infty)$	imaginary
Inverse of semicircle	$1/\sqrt{1 - x^2}$	$A = (-1,1)$	$f(A) = (0,1]$	real
	$i/\sqrt{x^2 - 1}$	$\tilde{A} = (-\infty, -1) \cup (1, \infty)$	$f(\tilde{A}) = (0, \infty)$	imaginary
Natural logarithm	$\ln x$	$A = (0, \infty)$	$f(A) = \mathbb{R}$	real
(main branch)	$\ln(-x) + i\pi$	$\tilde{A} = (-\infty, 0)$	$f(\tilde{A}) = \mathbb{R} \times \{\pi\}$	complex
Spindle	x^x	$A = A_n \cup [0, \infty)$	$f(A) = f(A_n) \cup [\mu, \infty)$	real
(main branch)	$e^{x[\ln(-x)+i\pi]}$	$\tilde{A} = A^c$	$f(\tilde{A}) = f(A^c)$	complex

4. Space Graphical Representation

The results relative to the range of the extended elementary functions given in Table 2, offer a new insight of how to realize a graphical representation, where real numbers obtained by applying function f on A may be displayed together with imaginary or complex numbers computed by evaluating f on $\tilde{A}$ (both processes comprise the extended function $\tilde{f}$). Therefore, we consider the Cartesian space xyz where xy corresponds to the ordinary real plane, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, formed by the coordinate x and y axes, and the vertical z axis (orthogonal to xy) is identified with the imaginary axis. Recall that in the geometrical representation of complex numbers, the real axis is usually denoted by Re whereas the imaginary axis is denoted by Im. Then, the "new" space, *G* = (Re x Re) x Im, can be shown to be the same as coordinate space $\mathbb{R}^3$. In this alternative interpretation, xz and yz are complex planes, i. e., equivalent to Re x Im. It is important to remark that the *extended graph* of $\tilde{f}$ will include, mainly, portions of the real xy plane, the complex xz plane, and, in some cases, portions of the semi-space defined by Re < 0. Also, all parallel planes to the xy plane are complex since they are either positive or negative translations relative to the imaginary axis z. Figures 1 to 6 show the graphs, in blue color (solid) of each elementary function f and its extended version $\tilde{f}$ as listed in Table 2.

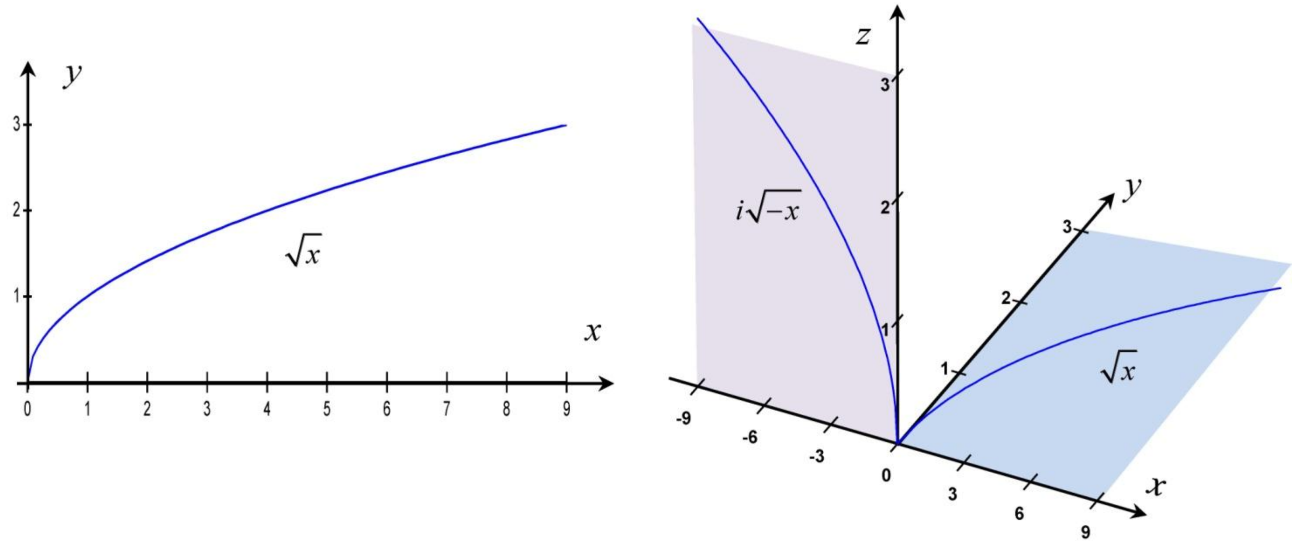

Fig. 1. Left: graph of the real “square root” function in the xy plane. Right: graph of its extended version in xyz space. In both cases, f and $\tilde{f}$ are depicted in blue color (solid curves). The z axis corresponds to the imaginary axis Im.

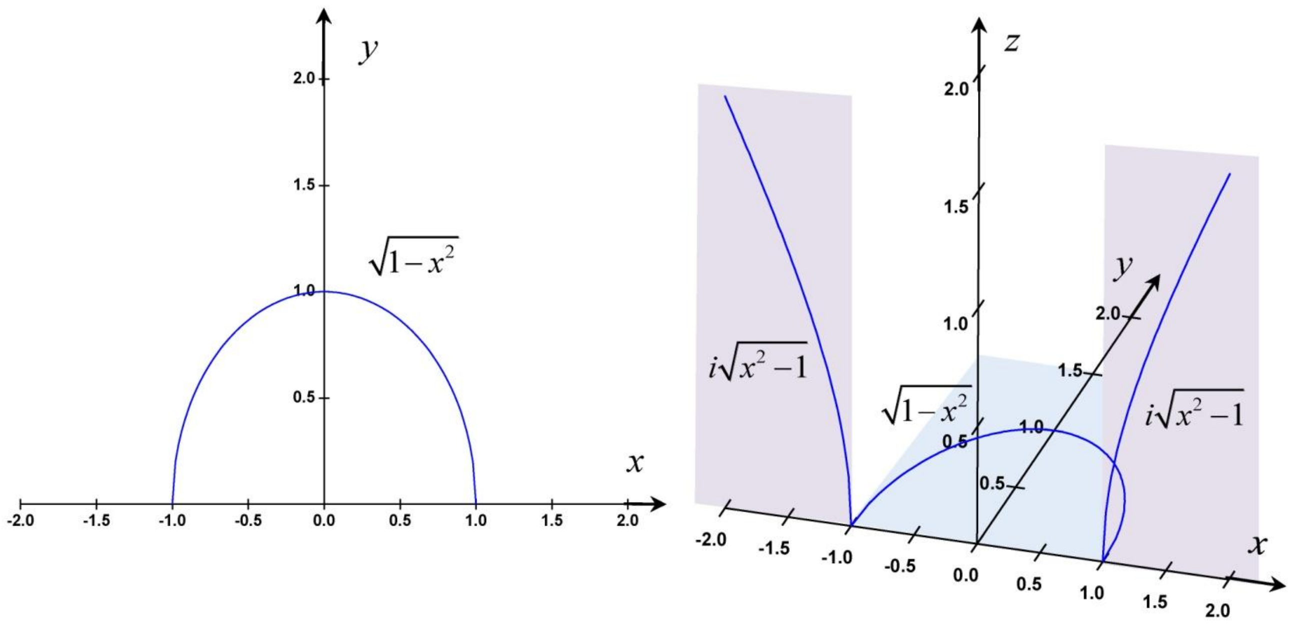

Fig. 2. Left: graph of the real “unit semicircle” function in the xy plane. Right: graph of its extended version in xyz space. In both cases, f and $\tilde{f}$ are depicted in blue color (solid curves). The z axis corresponds to the imaginary axis Im.

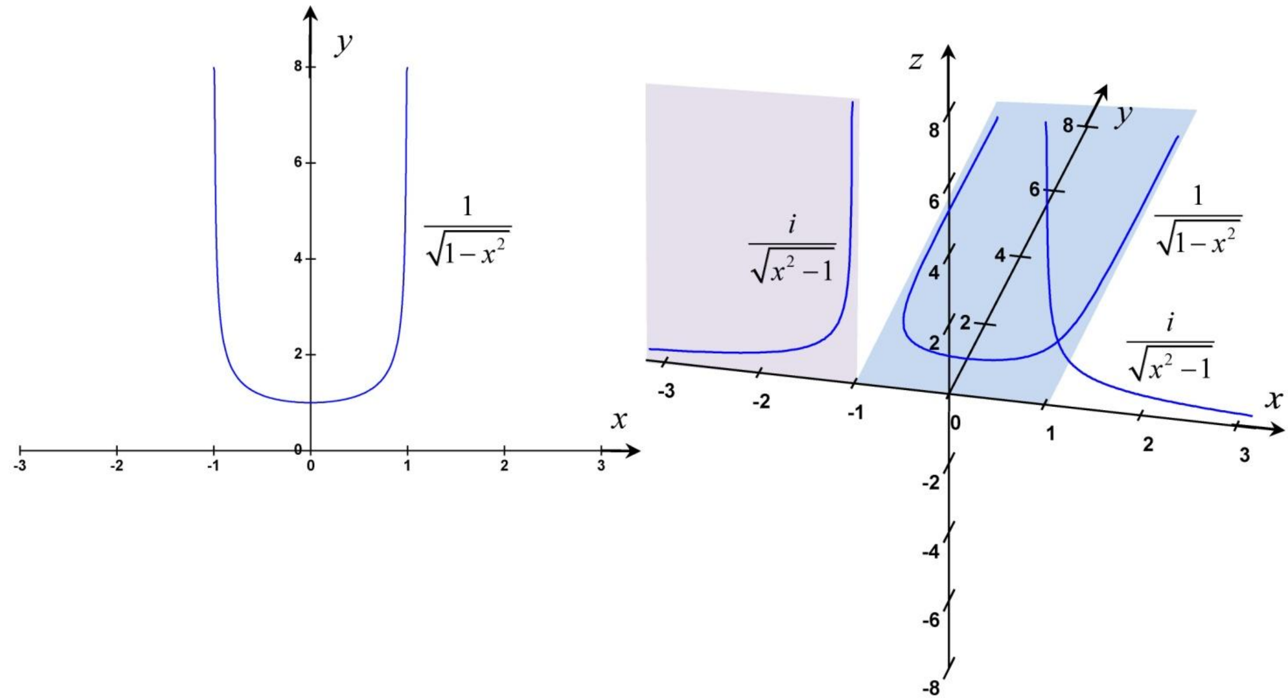

Fig. 3. Left: graph of the real "inverse of unit semicircle" function in the xy plane. Right: graph of its extended version in xyz space. In both cases, f and $\tilde{f}$ are depicted in blue color (solid curves).

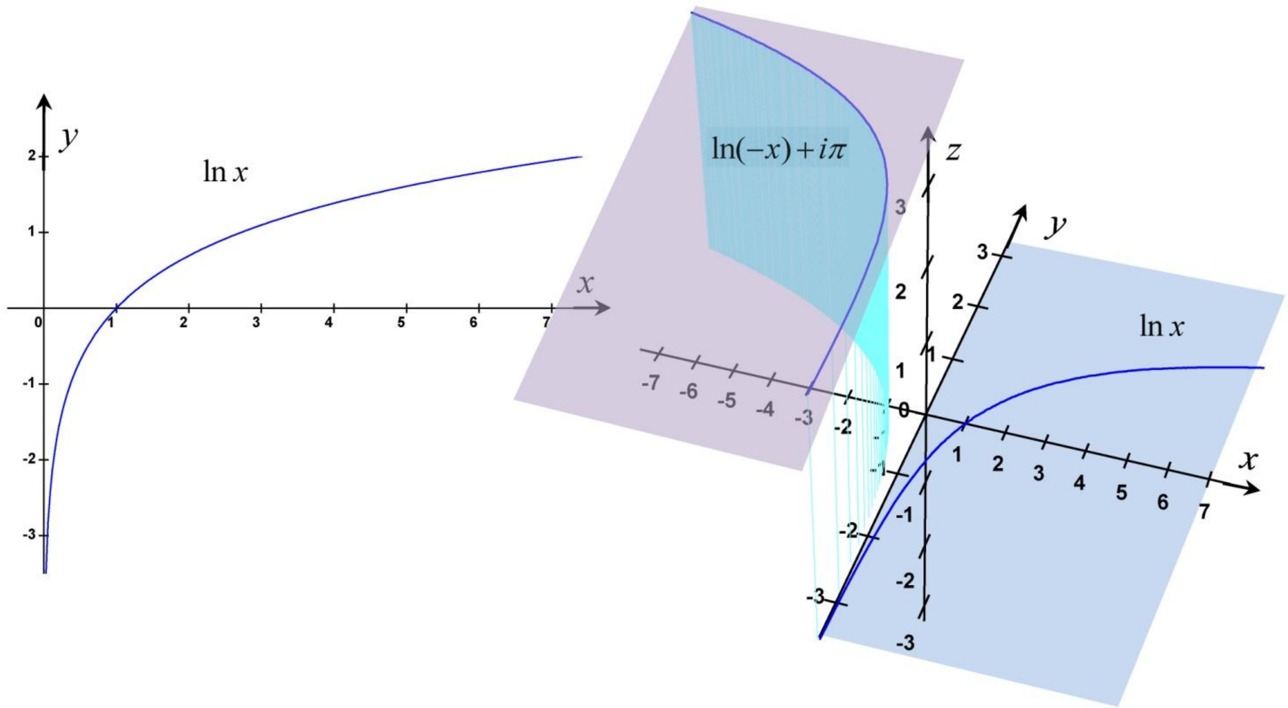

Fig. 4. Left: graph of the real "natural logarithm" function in the xy plane. Right: graph of its extended version in xyz space. In both cases, f and $\tilde{f}$ are depicted in blue color (solid curves). The vertical bands shown in light blue (cyan) emphasize the main branch upward projections of the complex values for $\ln x$ if $x < 0$ with respect to the xy plane.

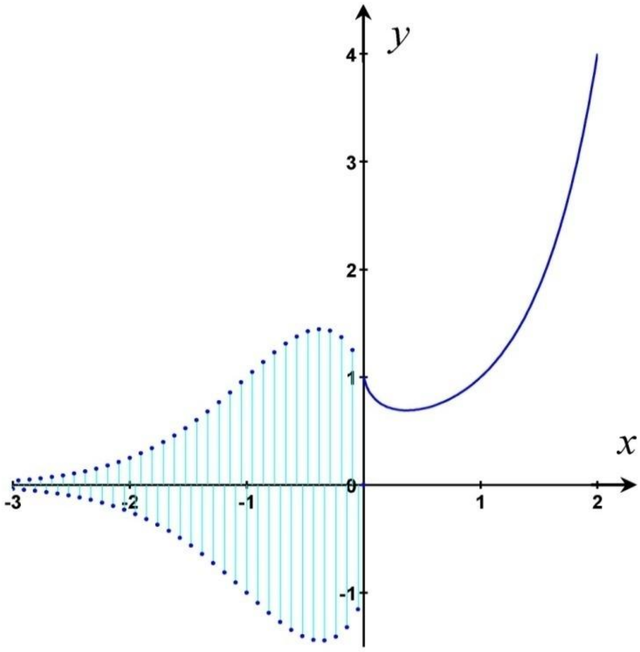

Fig. 5. Graph of the "spindle" elementary function. For, $x \geq 0$, the graph is continuous and for $x < 0$, the graph has many gaps and therefore is discontinuous. The 63 plotted points in the interval [-3,0) are integer multiples of the fraction 1/21 and the vertical lines projected onto the *x*-axis show how positive and negative *y* values are interlaced.

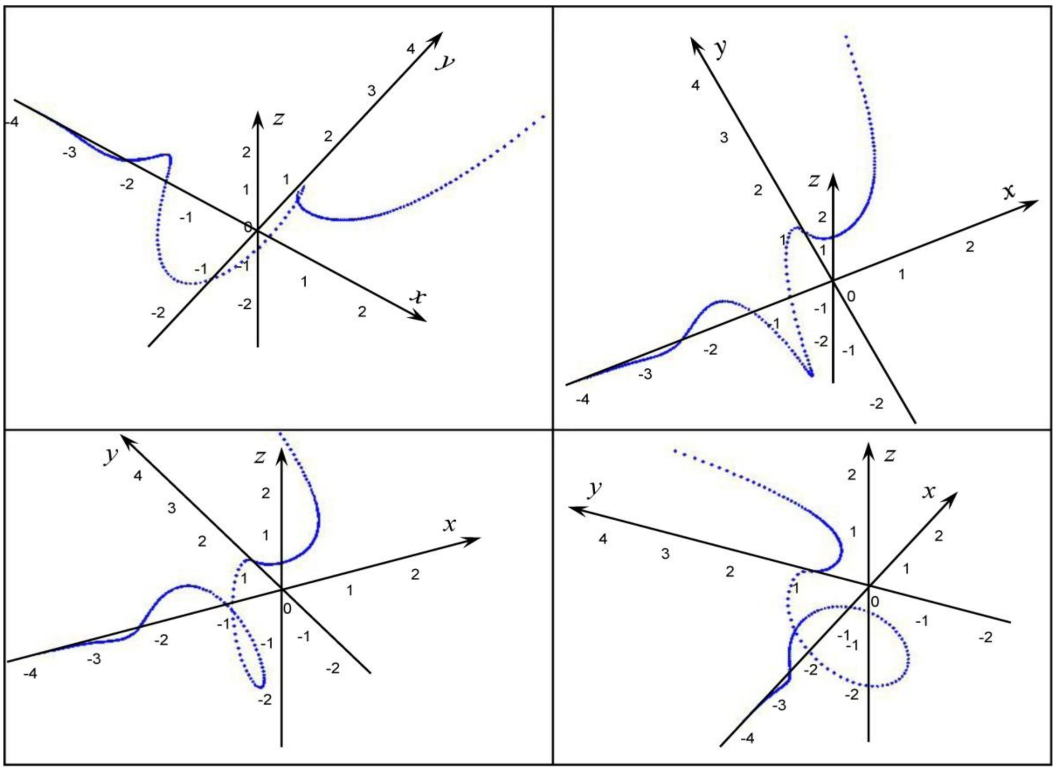

Fig. 6. Graph of the extended "spindle" elementary function. Four different views are displayed in xyz space that show the twist made by the principal branch in the interval (-3,0). Complex numbers belong to the semi-space Re < 0.

As can be seen from Table 1, the real polynomial functions can *not* be extended in the sense discussed earlier since their natural domain $A = \mathbb{R}$. However, the same spatial representation can be used to exhibit graphically the presence of complex roots. Previous efforts to visualize complex roots in the xy plane have been achieved for quadratic and cubic polynomials as reported, for example, in [22]-[25]. These approaches are based on simple algebraic analysis as well as the geometrical properties of the corresponding two-dimensional graph shapes but are not easily applied to any polynomial function. The main advantage of our graphical representation based on the G space is the direct and intuitive appeal in showing the coexistence of both real and complex roots.

For example, Fig. 7 shows five quadratic functions (parabolas) of the form, $y = x^2 + c$, where for each value of c, the ordinate at the origin (purple color), the *real roots* (in black color for $c = -4, -1, 0$), and the *conjugate complex roots* (in red color for $c = 1, 4$) are marked by small colored spheres that emphasize the relevance of these points. Analogously, Fig. 8 displays three cubic functions of the form, $y = x^3 + c$, that includes the distribution of all roots (real and complex) over the unit complex circle (shown in gray color) forming two equilateral triangles for $c = \pm 1$; clearly, if $c = 0$, the only triple real root of $x^3 = 0$ is $x = 0$ (the origin). Finally, Fig. 9 shows the graphs of five simple quartic functions given in Table 3. In this case, the quartic curves and their corresponding real or complex roots are drawn with the same color.

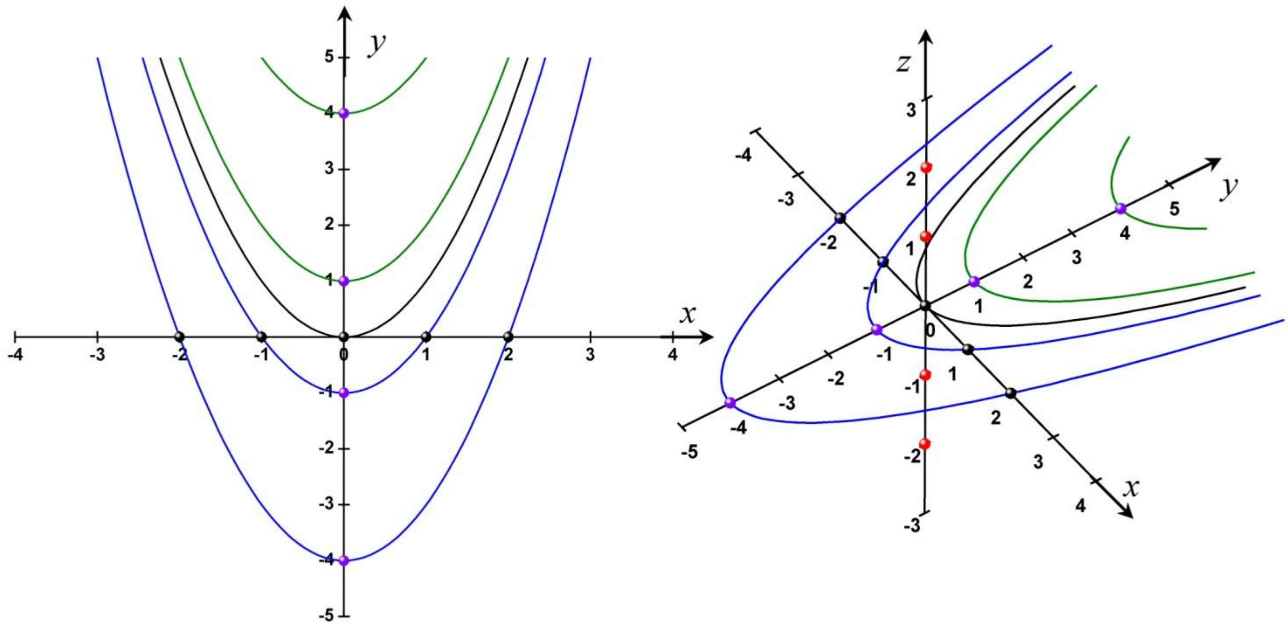

Fig. 7. Left: graphs of 5 "quadratic" elementary functions given by, $x^2 + c$ for $c = 0, \pm 1, \pm 4$, that show depending on the value of c, its real roots displayed as black dots (small spheres). Right: graphs in xyz space of the same "quadratic" elementary functions (non-extended) whose imaginary conjugate roots are shown on the imaginary z axis for $c = 1$ (roots are $\pm i$) and $c = 4$ (roots are $\pm 2i$) as red dots.

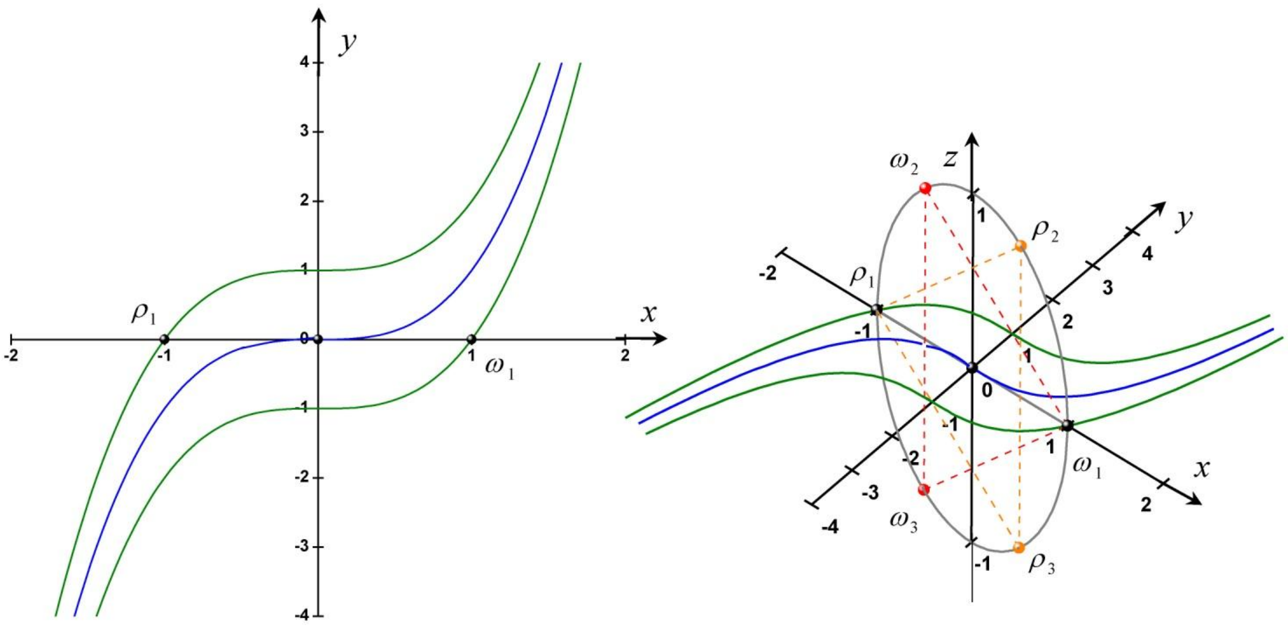

Fig. 8. Left: graphs of 3 “cubic” elementary functions given by, $x^3 + c$ for $c = 0, \pm 1$, that show depending on the value of c, its real roots displayed as black dots (small spheres). Right: graphs of the same “cubic” elementary functions (non-extended) whose complex roots are shown on the complex unit circle (xz plane) forming a pair of equilateral traingles, as orange dots for $c = -1$ and red dots for $c = 1$. The *x*-axis vertex of each triangle corresponds to a real root.

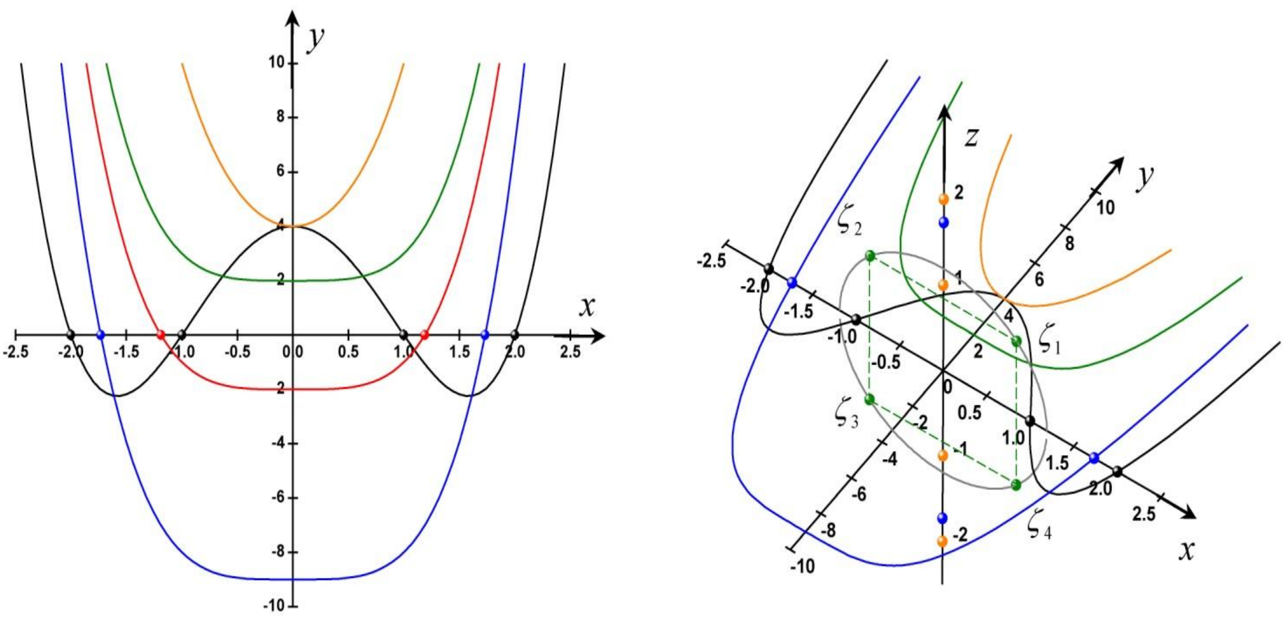

Fig. 9. Left: graphs of 5 “quartic” elementary functions that show their real roots (if any) displayed as colored small spheres. Right: graphs of the same “quartic” elementary functions (non-extended) whose complex roots are shown on the imaginary z axis or on the circle of radius $\sqrt[4]{2}$ forming a square on the xz complex plane (green dots).

The real, imaginary, and complex conjugate roots of the simple quadratic, cubic, and quartic polynomial type elementary functions are given explicitly in Table 3.

Table 3. Values of real, imaginary, or complex roots for polynomial functions of 2nd, 3rd, and 4th degree.

Assignment rule $y = f(x)$	**Roots**	**Type**
$x^2 - 4$	$x = \pm 2$	distinct real
$x^2 - 1$	$x = \pm 1$	distinct real
x^2	$x = 0$	equal real (double)
$x^2 + 1$	$x = \pm i$	conjugate imaginary
$x^2 + 4$	$x = \pm 2i$	conjugate imaginary
$x^3 + 1$	$\rho_1 = -1$ and $\rho_{2,3} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	real and complex conjugate
x^3	$x = 0$	equal real (triple)
$x^3 - 1$	$\omega_1 = 1$ and $\omega_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$	real and complex conjugate
$(x^2 - 4)(x^2 - 1)$	$x = \pm 1, \pm 2$	distinct real
$(x^2 - 3)(x^2 + 3)$	$x = \pm\sqrt{3}, \pm\sqrt{3}i$	distinct real; conjugate imaginary
$x^4 + 2$	$\zeta_k = \sqrt[4]{2} e^{i[0.785 + (k-1)\pi/2]}$	$k = 1, \ldots, 4$; complex conjugate
$(x^2 + 4)(x^2 + 1)$	$x = \pm i, \pm 2i$	conjugate imaginary

4. Conclusions

In this research work we have described a new graphical representation, in three dimensional space xyz, for functions of a real variable whose usual domain of definition can be extended to all the set of real numbers $\mathbb{R}$ (except for discontinuities) and with range a subset of the complex numbers $\mathbb{C}$. Some elementary functions of a real variable have been used as a guide to develop naively the concept of extended elementary function from which, by geometrical interpretation, the concept of a G space equivalent to $\mathbb{R}^3$ provides the framework to visualize the associated extended graphs. The advantage of the proposed graphical representation, in comparison with the traditional point of view, resides in being able to assimilate directly and in a natural way the coexistence of real as well as complex numerical values in a single scenario. Additionally, the same graphical representation allows one to see the presence of conjugate complex roots of real polynomial functions in relation to the fundamental theorem of algebra. The graphs displayed in Figs. 1 to 9 where prepared with the following software packages: Mathcad (computation for data generation in xyz space) [26], Origin (3D curve tracing by layers) [27], Geogebra (3D views by rotation) [28] and PowerPoint (vectorial composition, retouch, and mathematical labeling).

Future work contemplates the realization of additional graphs of other elementary functions of a real variable, susceptible to be extended in the sense defined here, justification in greater detail of the algebraic structure underlying the G space, and to make available this type of graphical representation as a didactic tool to support comprehension in certain topics of basic algebra, analytic geometry, and calculus.

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References

[1] Nicholas J., Hunter J., & Hargreaves J. (1997), *Functions and Their Graphs*, Mathematics Learning Center, University of Sidney, Sidney, Australia, 1-74.

[2] Bourne M. (2017), Interactive Mathematics: Learn math while you play with it (Site Info), Singapore, https://www.intmath.com/help/site-info.php

[3] Courant R. & Robbins H. (1996), Functions and Limits. In *What is Mathematics? An Elementary Approach to Ideas and Methods, 2nd Ed.*, Oxford University Press, New York, Ch. VI, 272-289.

[4] Spivak M. (1994), Functions and Graphs. In *Calculus, 3rd Ed.*, Publish or Perish, Houston, Texas, Ch. 3, 39-55 and Ch. 4, 56-74.

[5] Murrill M.L. (1959), *Four Dimensional Graphs of Complex Functions*, Master Thesis, University of Richmond, Virginia, USA.

[6] Thaller B. (1998), Visualization of Complex Functions, *The Mathematica Journal* 7(2), 163-180.

[7] Liebo R. (2006), *Visualization of Complex Functions Graphs in Augmented Reality*, Master Thesis, Vienna University of Technology, Vienna, Austria.

[8] Kaufmann H. (2009), Virtual Environments for Mathematics and Geometry Education, *Themes in Science and Technology Education*, Special Issue, Klidarithmos Computer Books, 131-152.

[9] Eisenberg M. & Park D.J.M. Jr. (2009), Visualizing Complex Functions with the Presentations Application, *The Mathematica Journal* 11(2), 226-251.

[10] Avitzur R., Gooding A., Herrmann E., Piovanelli M., et al. (2002), *Learning Math with Graphing Calculator: Investigating Symbols, Functions, and Graphs*, Pacific Technologies, Oakland, California.

[11] Pacific Tech (2004), *Graphing Calculator*, Berkeley, California, https://www.pacifict.com

[12] Douglas N.A. (1997), Graphics for Complex Analysis, School of Mathematics, University of Minnesota, http://www-users.math.umn.edu/~arnold/complex.html

[13] Lundmark H. (2004), Visualizing complex analytic functions using domain coloring, Linköping Univer-sity, Sweden, https://web.archive.org/web/20060502154939/http://www.mai.liu.se/~halun/complex/domain_coloring-unicode.html

[14] Wikipedia (2017), Domain Coloring, https://en.wikipedia.org/wiki/Domain_coloring

[15] Blake S. (2011), Plotting Functions and Graphs in Wolfram Alpha, *WolframAlpha Blog Archive*. http://blog.wolframalpha.com/2011/05/17/plotting-functions-and-graphs-in-wolframalpha/

[16] MathWorks (2017), Easy Plotting: Graphs of Functions, *R2017b Documentation MATLAB/ MuPAD*. https://www.mathworks.com/help/symbolic/mupad_ug/easy-plotting-graphs-of functions.html?requestedDomain=www.mathworks.com&requestedDomain= www.mathworks.com

[17] The Sage Development Team (2017), 2D Graphics and 3D Graphics, *Sage Reference Manual*, http://doc.sagemath.org/html/en/reference/index.html

[18] Runiter Company (2017), Graphing Calculator 3D, Visualize Math and Data, https://www.runiter.com/graphing-calculator/

[19] Wikipedia (2017), Tetration, https://en.wikipedia.org/wiki/Tetration

[20] Wolfram MathWorld (2017), Power Tower, http://mathworld.wolfram.com/PowerTower.html

[21] Meyerson M.D. (1996), The x^x Spindle, *Mathematics Magazine* 69(3), 198-206.

[22] Norton A. & Lotto B. (1984), Complex roots made visible, *The College Mathematics Journal* 15(3), 248-249.

[23] Su F.E., et al. (2010), Complex roots made visible, *Math Fun Facts*, https://www.math.hmc.edu/funfacts/ffiles/10005.1.shtml

[24] British International School-Phuket, Graphically Understanding Complex Roots, IB Maths Resources https://ibmathsresources.com/2014/04/25/graphically-understanding-complex-roots/

[25] Orsolya A. P-S. (2015), Where are the quadratic's complex roots?, *Acta Didactica Napocensia* 8(1), 37-47.

[26] Parametric Technologies Corporation (2017), *PTC Mathcad, Engineering Mathematical Software-Mathcad Prime 4.0*, https://www.ptc.es/engineering-math-software/mathcad

[27] OriginLab Corporation (2017), *Origin 2017-Graphing and Analysis*, www.originlab.com

[28] Hohenwarter M., et al. (2017), *Geogebra – Dynamic Mathematics for Everyone*, http://www.geogebra.org/