Convergence properties of linear k −positive operators in subspace of entire functions

SEDA KARATEKE [1] and ERDAL COŞKUN [2]

[1] Istanbul Arel University, Faculty of Science and Letters, Department of Mathematics and Computer Science, 34537 Tepekent-Istanbul, TURKEY
Email: sedakarateke34@gmail.com

[2] Bülent Ecevit University, Faculty of Science and Letters, Department of Mathematics, Zonguldak, TURKEY
Email: aerdal.coskun@gmail.com

Abstract:
In this article; firstly, we consider an application with graphs and tables that puts in evidence that Korovkin-type convergence conditions are not valid for the space of entire functions on the unit disk of complex plane denoted by A. Then, we prove that the convergence case is positive for a proper subspace of A for linear $k-$ positive operators. At the end of this study, we visualize the approximation case on above-mentioned subspace by tackling an example by means of Maple.

Key-Words: Korovkin-type theorem; k-positive linear operators; analytic functions; test functions; analytic space; positive linear operators

1. Introduction

Approximation theory has an essential role in literature (see [1], [2], [5], [7], [9]). Some mathematicians have studied the approximation of analytic functions by the sequences of linear operators by means of the ones which have k-positivity features. The concept of k −positiveness was introduced in [4] by A. D. Gadjiev. He developed a technique for solving problems of this field. Miscellaneous works have been tackled in [11-25]. A work without k −positivity was considered in [22]. Korovkin's theorem provides a very useful and simple criterion for $(T_n)_{n\geq 1}$ of positive linear operators on $C[a,b]$, it is an approximation operation, i.e., $T_n(t^k;x), k = 0,1,2$ uniformly on $[a,b]$ for each functions $f(x) \in C[a,b]$, where $T_n: C[a,b] \to C[a,b]$ (see [2], [9]). In the articles [4] and [10], A.D. Gadjiev gave the space of entire functions defined on open domain of complex plane, denoted by A to A and proposed the Korovkin type theorems on convergence of these operators in different subspaces of A. A comprehensive picture of what has been achieved in this field is documented in the monographs [8,11,12,14,15,16,17,19,20,21]. Furthermore, Pai and Jain studied on Korovkin's type approximation for complex-valued functions on compact subset (see[6]). Papers [13,14,15,16] are considerable for different types of convergence via aforesaid operators. The article [21] refers to the space of analytic functions in closed domain and paper [23] is related to the convergence of entire functions in anulus by linear operators. Articles [24] and [25] perform theorems for linear $k-$positive operators of analytic functions in polydiscs. Recent works concerning linear $k-$positive operators motivated us to study the convergence on the of these operators in different subspaces of A. Main goal in our work is to analyze approximation of analytic functions in convenient subspaces by sequences of linear $k-$positive operators. We give necessary and sufficient conditions in some subspace of A. In addition to this we compare and visualize the convergence case between the space A and subspace A_g which are enriched with example, numerical results and graphs by means of Maple.

2 Preliminaries

$\mathbb{N}$ and $\mathbb{C}$ are the set of natural numbers and the space of complex numbers, sequentially and $D_0 = \{z \in \mathbb{C} : |z| < 1\}$ is a unit disc. In space A, we mean uniform convergence in any closed domain of D_0. If $f \in A$ then Taylor series of function $f \in A$ is represented as follows:

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \qquad (2.1)$$

where f_k is the Taylor coefficients of $f(z)$, and $\overline{\lim_{k\to\infty}} \sqrt[k]{|f_k|} = 1$. For any fixed $r < 1$, (2.1) is convergent uniformly in any closed subdisks $|z| \le r < 1$.

$$\|f\|_A = \max_{|z|\le r<1} |f(z)|$$

defines a seminorm in the space of analytic functions A. A becomes a *Fréchet* type space for $0 < r < 1$ according to aforementioned convergence. Taylor expansion of the sequence of functions $\left(f_n(z)\right) \in A$ is defined as follows

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k} z^k$$

where $f_{n,k}$ is Taylor coefficients of $f_n(z)$ and for any real number $\rho < 1$:

$$f_{n,k} = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f_n(z)}{z^{k+1}} dz$$

Linear $T: A \to A$ is defined as "$k-$positive" if T transforms analytic functions whose coefficients are nonnegative to the same type functions. It is obvious to see that any linear operator acting from A to A can be represented in the form

$$Tf(z) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m} f_m$$

where $T_{k,m}$ is the some matrix of numbers, connected with the operator T such that

$$\overline{\lim_{k\to\infty}} \left| \sum_{m=0}^{\infty} T_{k,m} f_m \right|^{\frac{1}{k}} = 1.$$

We will study the sequence of linear operators

$$T_n f(z) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} f_m$$

acting on functions $f \in A$ having the qualifications of "k −positivity" (see [3], [4], [10], and [26]).

Lemma 2.1 (see [10])

Sequences

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k}\, z^k$$

converges to zero if and only if the following conditions i) ve ii) are valid.

i) There exists some sequence of positive numbers $\varepsilon'_{n,k}$ verifying inequality $|f_{n,k}| \leq \varepsilon'_{n,k}$ such that $\lim_{n\to\infty} \varepsilon'_{n,k} = 0$ for fixed number k, and every $n = 0,1,2, \ldots$

ii) There exists some sequence of positive numbers $\varepsilon''_{n,k}$ satisfying the following inequalities

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\varepsilon''_{n,k}} = 1, \qquad |f_{n,k}| \leq \varepsilon''_{n,k}$$

for every $k = 0,1,2, \ldots$ and fixed number n.

3 Convergence in sense of *k*-positivity in *A*

Theorem 3.1 ([10]) If sequences of k-positive linear operators $T_n: A \to A$ verify following condition

$$\lim_{n\to\infty} \left\| T_n z^\vartheta - z^\vartheta \right\|_A = 0 \tag{3.1}$$

for every $n \in \mathbb{N}$ and $\vartheta = 0,1,2, \ldots$ Then for function $\frac{1}{1-z} \in A$

$$\lim_{n\to\infty} \left\| T_n \frac{1}{1-z} - \frac{1}{1-z} \right\|_A \neq 0$$

Remark 3.1 Now, we tackle an application of above theorem. Our aim is to see that Korovkin-type approximation in the space A is not valid, by using numerical results and graphs drawn by means of Maple. So, we will need to take subspace to get positive results in the sense of Korovkin-type convergence (see below Theorem 4.1). Let the functions z^ϑ satify the condition (3.1) for $= 0,1,2, \ldots$. Then for the linear k−positive operator

$$T_n f(z) = \sum_{k=0}^{\infty} f_k \left[z^k + \frac{1}{n} \left(\frac{n+z}{n+1} \right)^k \right]$$

, and the below function $f(z)$ which belongs to space A for $|z| < 1$

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

the Korovkin-type convergence in the space A is not provided (see the Table 3.1 for numerical results in Appendices).

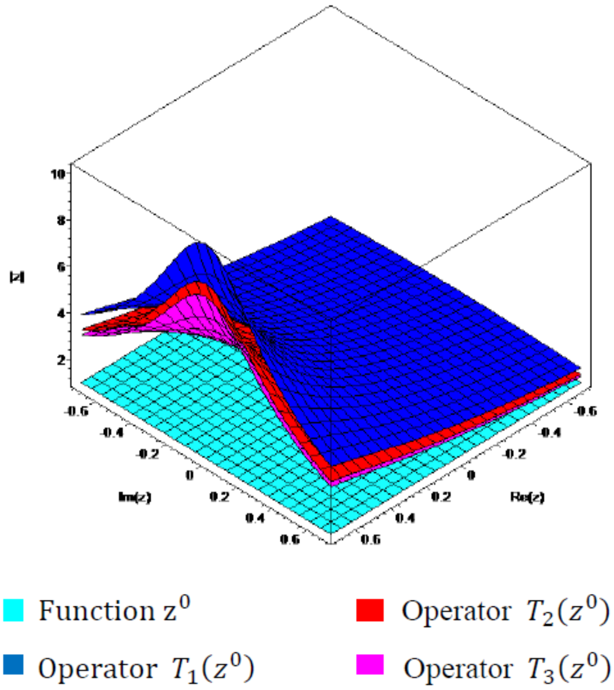


Figure 3.1: Approximation to test function z^0 for $n = 1, n = 2$ and $n = 3$.

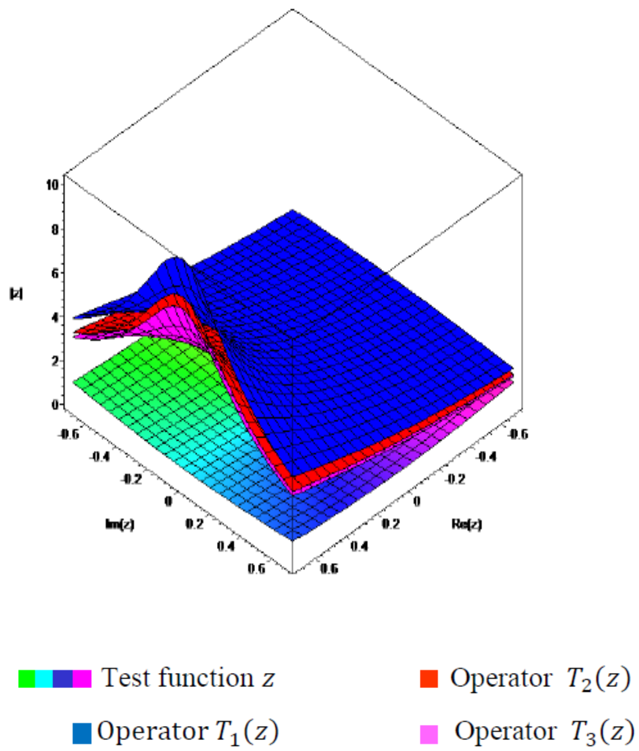


Figure 3.2: Approximation to test function z for $n = 1, n = 2$ and $n = 3$.

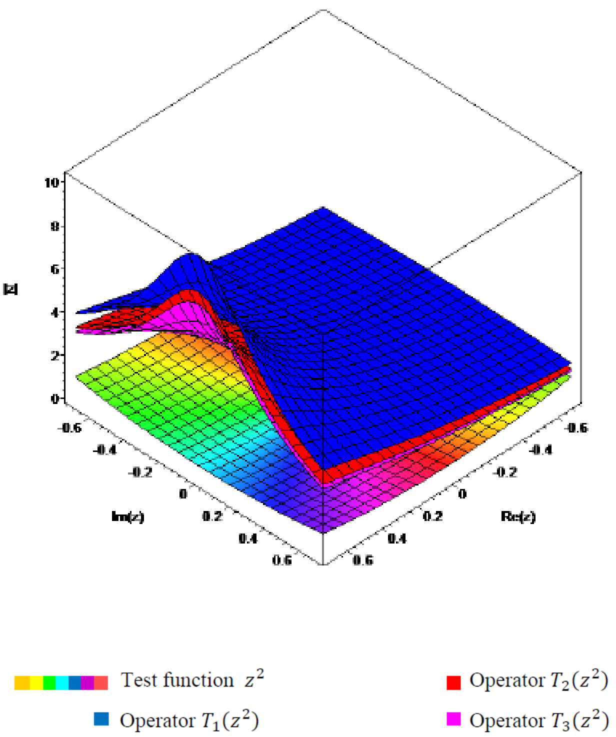


Figure 3.3 : Approximation to test function z^2 for $n = 1, n = 2$ and $n = 3$.

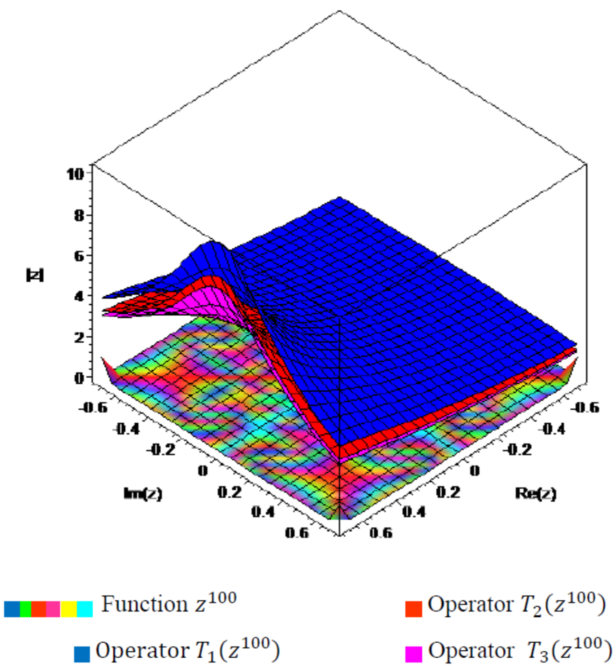
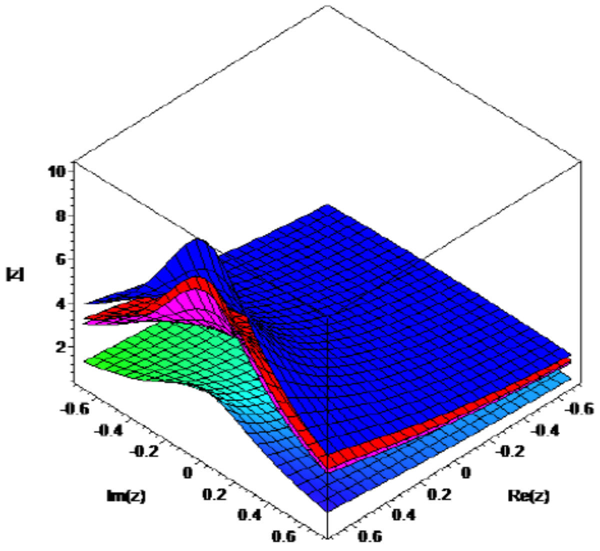


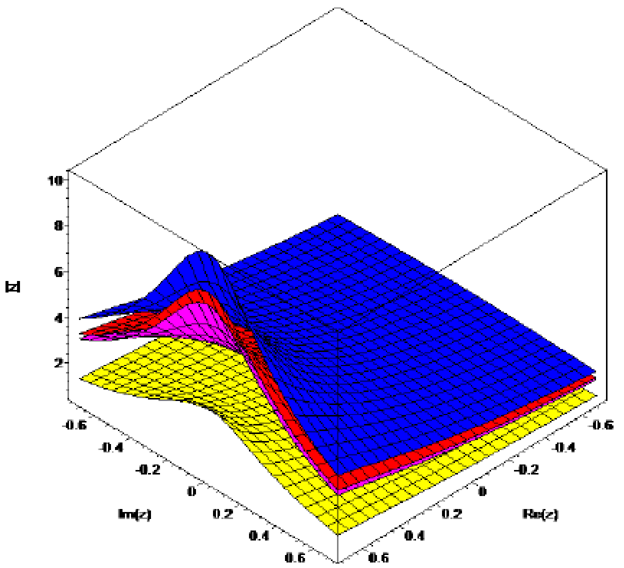
Figure 3.4: Approximation to function z^{100} for $n = 1, n = 2$ and $n = 3$.



Function $f(z) = \frac{1}{1-z}$ Operator $T_2(f(z))$

Operator $T_1(f(z))$ Operator $T_3(f(z))$

Figure 3.5:Approximation to function $f(z) = \frac{1}{1-z}$ **for** $\boldsymbol{n = 1, n = 2}$ **and** $\boldsymbol{n = 3}$**.**



Series $\sum_{k=0}^{\infty} z^k$ Operator $T_2(f(z))$

Operator $T_1(f(z))$ Operator $T_3(f(z))$

Figure 3.6: Approximation to series $\sum_{k=0}^{\infty} z^k$ **for** $\boldsymbol{n = 1, n = 2}$ **and** $\boldsymbol{n = 3}$**.**

3 Korovkin type convergence conditions in sense of k-positivity for subspace A_g

We now consider the following subspace of A:

Let $g:[0,\infty[\to[0,\infty[$, $g(k)=\left(1+k^{(2p)}\right)$ monotone increasing function such that $g(0)=1$ and $\overline{\lim_{k\to\infty}}\sqrt[k]{(1+k^{2p})}=1$

$$A_g:=\left\{f\in A:|f_k|\le M_f(1+k^{2p}),k=0,1,2,\dots\right\}$$

where M_f is some constant that is depend on f and $p\in\mathbb{N}_0$, the set of natural numbers.

Theorem 4.1 Let $T_n:A_g\to A$ be the sequences of linear $k-$positive operators, in order

$$\lim_{n\to\infty}\|T_nf-f\|_A=0$$

be true for every function $f\in A_g$, it is necessary and sufficient that

$$\lim_{n\to\infty}\|T_n\tilde{g}_\vartheta(z)-\tilde{g}_\vartheta(z)\|_A=0,\qquad \vartheta=0,1,2$$

where

$$\tilde{g}_\vartheta(z)=\sum_{k=0}^{\infty}k^{p\vartheta}\,z^k\,,\quad \vartheta=0,1,2$$

Proof.

$$T_nf(z)-f(z)=\sum_{k=0}^{\infty}z^k\sum_{m=0}^{\infty}f_mT_{k,m}^{(n)}-\sum_{k=0}^{\infty}f_kz^k=\sum_{k=0}^{\infty}z^k\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,[f_m-f_k]+\sum_{k=0}^{\infty}f_kz^k\left[\sum_{m=0}^{\infty}T_{k,m}^{(n)}-1\right]$$

and by taking absolute value of all terms,

$$|T_nf(z)-f(z)|\le\sum_{k=0}^{\infty}|z|^k\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,|f_m-f_k|+\sum_{k=0}^{\infty}|z|^k\,|f_k|\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}-1\right|$$

Since $f\in A_g$ condition $|f_k|\le M_f(1+k^{2p})$ is satisfied.

Let us put that

$$S'_{n,k}\;=\sum_{k=0}^{\infty}|z|^k\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,|f_m-f_k|$$

and

$$S''_{n,k}\;=\sum_{k=0}^{\infty}|z|^k\,|f_k|\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}-1\right|.$$

Here, due to Lemma 2.1 it is sufficient for the proof to show that the sequences $S'_{n,k}$ and $S''_{n,k}$ belong to the space $E_{n,k}$.

$m,k\in\mathbb{N}$; for $m<k$, there exists the inequalities $m-k<k$ and $m^p-k^p<k^p$

$$\begin{aligned}|f_m-f_k|&\le|f_m|+|f_k|\\&\le M_f(1+m^{2p}+1+k^{2p})\\&\le 4M_f(1+(m^p-k^p)^2+k^{2p})\end{aligned}$$

and since for $m\neq k$ we know that the inequality $(m-k)^2\ge1$ is true

$|f_m-f_k|\;\le 8M_f(m^p-k^p)^2(1+k^{2p})$

Furthermore

$$T_n\tilde{g}_\vartheta(z)-\tilde{g}_\vartheta(z)=\sum_{k=0}^{\infty}z^k\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,m^{p\vartheta}-\sum_{k=0}^{\infty}k^{p\vartheta}z^k$$

$$=\sum_{k=0}^{\infty}z^k\left(\sum_{m=0}^{\infty}T_{k,m}^{(n)}\;m^{p\vartheta}-k^{p\vartheta}\right)$$

is valid due to Lemma 2.1 for $\vartheta = 0,1,2$

$$\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,m^{p\vartheta}-k^{p\vartheta}\right|<\varepsilon_{n,k}$$

such that $\varepsilon_{n,k}\in E_{n,k}$.

For $= 0$,

$$\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}-1\right|<\varepsilon_{n,k}$$

For $= 1$,

$$\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,m^{p}-k^{p}\right|<\varepsilon_{n,k}$$

For $\vartheta = 2$,

$$\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,m^{2p}-k^{2p}\right|<\varepsilon_{n,k}$$

If the above inequalities are rearrenged,

$$\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,k^{2p}-\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,2\,k^pm^p+\sum_{m=0}^{\infty}\varepsilon_{n,k}\,m^{2p}<(k^{2p}+2k^p+1)\varepsilon_{n,k}$$

$$\sum_{m=0}^{\infty}T_{k,m}^{(n)}(k^p-m^p)^2<(1+k^p)^2\,\varepsilon_{n,k}$$

Thus for $m<k$

$$S'_{n,k}\coloneqq\sum_{m=0}^{\infty}T_{k,m}^{(n)}\,|f_m-f_k|$$

$$<8\,M_f(1+k^{2p})(1+k^{2p})^2\varepsilon_{n,k}=\varepsilon'_{n,k}$$

and

since the equalities $\overline{\lim_{k\to\infty}}\sqrt[k]{\varepsilon_{n,k}}=1$ and $\overline{\lim_{k\to\infty}}\sqrt[k]{(1+k^{2p})^3}=1$ exist, the equality $\overline{\lim_{k\to\infty}}\sqrt[k]{\varepsilon'_{n,k}}=1$ exists, too. Thus

$$\lim_{n\to\infty}S'_{n,k}=0$$

is obtained.

Now the below inequality is satisfied, too.

$$S''_{n,k}=\sum_{k=0}^{\infty}|z|^k|f_k|\left|\sum_{m=0}^{\infty}T_{k,m}^{(n)}-1\right|$$

$$\leq \sum_{k=0}^{\infty} |z|^k (1 + k^{2p})\, \varepsilon_{n,k}$$

and

$$\lim_{n\to\infty} S''_{n,k} = 0.$$

Since the following inclusion exists

$$\left(S'_{n,k} + S''_{n,k}\right) \in E_{n,k}$$

$\lim_{n\to\infty} \|T_n f - f\|_A = 0.$

On the other hand, let us accept that the following convergence is valid for every $f \in A_g$

$$\lim_{n\to\infty} \|T_n f - f\|_{A_g} = 0$$

From here

for $= 0$,

$$\tilde{g}_0(z) = \sum_{k=0}^{\infty} z^k$$

for $= 1$,

$$\tilde{g}_1(z) = \sum_{k=0}^{\infty} k^p z^k$$

for $= 2$,

$$\tilde{g}_2(z) = \sum_{k=0}^{\infty} k^{2p} z^k$$

There exists $M_f = 1$ that satisfies the following inequalities for the coefficients of these kinds of functions

$|1| \leq M_f (1 + k^{2p})$

$|k^p| \leq M_f (1 + k^{2p})$

$|k^{2p}| \leq M_f (1 + k^{2p})$

This means that $\tilde{g}_\vartheta(z) \in A_g$ for $= 0, 1, 2$. Thus

$$\lim_{n\to\infty} \|T_n \tilde{g}_\vartheta(z) - \tilde{g}_\vartheta(z)\|_A = 0.$$

and proof has been completed.

Example 4.1

For test function $\tilde{g}_\vartheta(z) = \sum_{k=0}^{\infty} k^\vartheta z^k$, $(\vartheta = 0,1,2)$ and linear $k-$positive operator

$T_n\big(f(z)\big) = \sum_{k=0}^{\infty} f_k \left[z^k + \frac{1}{n^2}\left(\frac{n+z}{n+1}\right)^k\right]$

and for $|z| < 1$ the function

$$f(z) = e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k,$$

the following equality

$$\lim_{n\to\infty} \|T_n f(z) - f(z)\|_A = 0$$

is valid.

To see that this convergence state is valid for the chosen test functions, operator and function due to Theorem 4.1 for $\vartheta = 0, 1, 2$ the below equality

$$\lim_{n\to\infty}\|T_n\tilde{g}_\vartheta(z)-\tilde{g}_\vartheta(z)\|_A=0$$

is verified.
For $\vartheta=0$

$$\tilde{g}_0(z)=\sum_{k=0}^{\infty}z^k$$

$$T_n\big(\tilde{g}_0(z)\big)=\tilde{g}_0(z)+\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{1}{k!}z^k$$

and since the series

$$\sum_{k=0}^{\infty}\frac{1}{k!}z^k$$

is convergent for $|z|<1$

$$\lim_{n\to\infty}T_n\big(\tilde{g}_0(z)\big)=\tilde{g}_0(z)$$

is true.
For $\vartheta=1$

$$\tilde{g}_1(z)=\sum_{k=0}^{\infty}k\,z^k$$

$$T_n\big(\tilde{g}_1(z)\big)=\tilde{g}_1(z)+\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{k}{k!}z^k$$

and the series

$$\sum_{k=0}^{\infty}\frac{k}{k!}z^k$$

is convergent for $|z|<1$

$$\lim_{n\to\infty}T_n\big(\tilde{g}_1(z)\big)=\tilde{g}_1(z)$$

is true. For $\vartheta=2$

$$\tilde{g}_2(z)=\sum_{k=0}^{\infty}k^2z^k$$

$$T_n\big(\tilde{g}_2(z)\big)=\tilde{g}_2(z)+\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{k^2}{k!}z^k$$

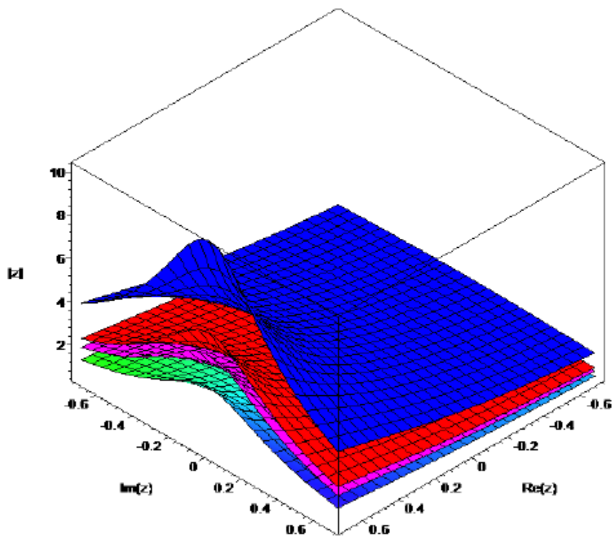
and the series

$$\sum_{k=0}^{\infty}\frac{k^2}{k!}z^k$$

is convergent for $|z|<1$

$$\lim_{n\to\infty}T_n\big(\tilde{g}_2(z)\big)=\tilde{g}_2(z)$$

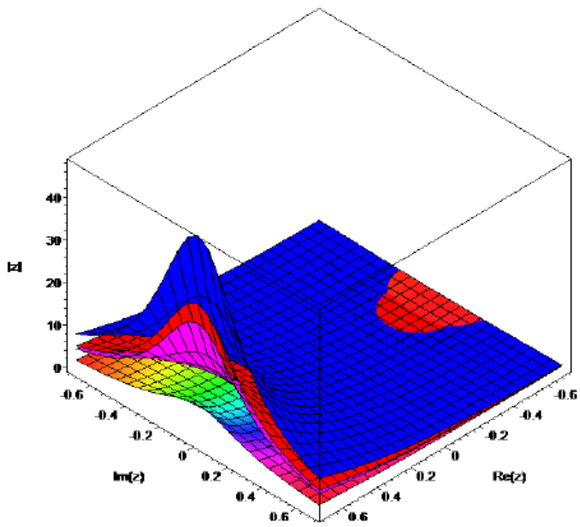
is true, too. Approximation to the function $f(z)=e^z$ for some $|z|<1$ by the linear k −positive operators T_n is shown by using Maple as follows (see the Table 4.1 for numerical results in Appendices).



Test function $\tilde{g}_0(z) = \sum_{k=0}^{\infty} z^k$ Operator $T_2(\tilde{g}_0)$

Operator $T_1(\tilde{g}_0)$ Operator $T_3(\tilde{g}_0)$

Figure 4.1: Approximation to the test function $\tilde{g}_0(z) = \sum_{k=0}^{\infty} z^k$ for $n = 1, n = 2$ and $n = 3$



Test function $\tilde{g}_1(z) = \sum_{k=0}^{\infty} k\, z^k$ Operator $T_2(\tilde{g}_1)$

Operator $T_1(\tilde{g}_1(z))$ Operator $T_3(\tilde{g}_1)$

Figure 4.2: Approximation to the test function $\tilde{g}_0(z) = \sum_{k=0}^{\infty} kz^k$ for $n = 1, n = 2$ and $n = 3$.

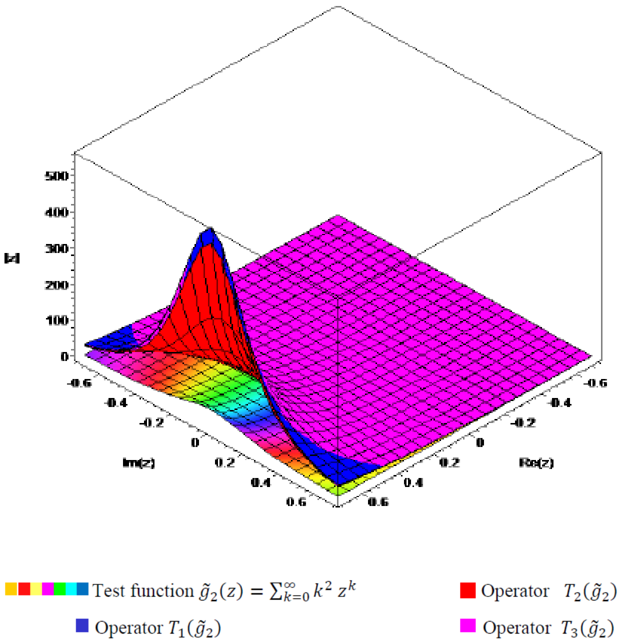


Figure 4.3: Approximation to the test function $\tilde{g}_0(z) = \sum_{k=0}^{\infty} k^2 z^k$ for $n = 1, n = 2$ and $n = 3$.

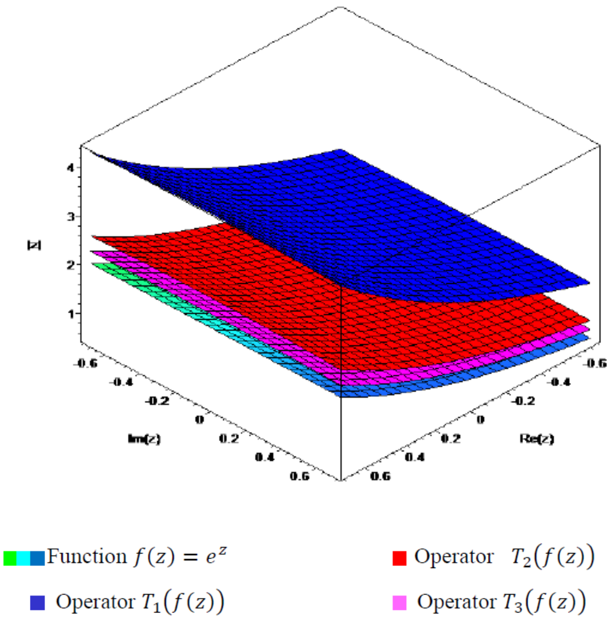


Figure 4.4: Approximation to the function $\boldsymbol{f(z) = e^z}$ for $n = 1, n = 2$ and $n = 3$.

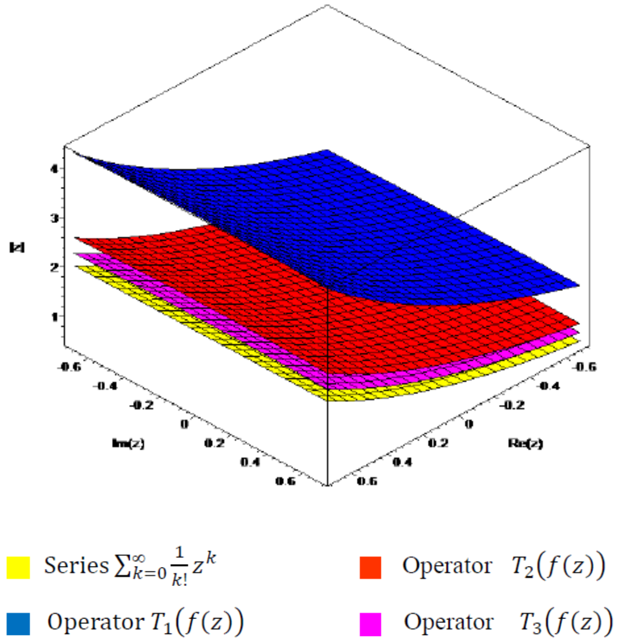


Figure 4.5: Approximation to the series $\sum_{k=0}^{\infty}\frac{1}{k!}z^k$ for $n = 1, n = 2$ and $n = 3$.

4 Results and discussion

As it is seen above Theorem 3.1 ([1]) and Table 3.1 ; for the countable functions z^k; $k = 0,1,2,...$ the linear k-positive operators do not, unfortunately, converge to arbitrary functions in the space A. On the other hand, Theorem 4.1 and Table 4.1 indicate that the convergence state that is not valid for the space A is valid for the subspace A_g.

5 Conclusions

We obtain some approximation results for convenient subspace of A in Gadjiev sense. By comparing approximation results between in spaces A and A_g using convenient functions and operators, we get numerical results which are supported by graphs by means of Maple.

References:

[1] Khaplanov M., Linear operators in analytic space, *Rostov. Gos. Univ. Uch. Zap. Fiz.-Mat. Fak.* 43 (1959) 83-118 (in Russian).

[2] Korovkin P.P.: Linear operators and Approximation Theory, Translated from the Russian ed. (1959), *Russian Monographs and Texts on advances Mathematics and Physics, Vol. III, Gordon and Breach Publishers, Ins. New York, Hindustan Publ. Corb.* (India), Delhi, 1960

[3] Evgrafov M.A., *The method of near systems in the sapce of analytic functions and its application to interpolation*, Tr. Mosk. Mat. Obs. 5 (1956) 89-201 (in Russian); Amer. Math. Soc. Transl. 16 (1960) 195-314 (English transi.)

[4] Gadjiev A. D., Linear k-positive operators in a space of regular functions, and theorems of P.P. Korovkin type, *Izv. Akad. Nauk Azerb. SSR Ser. Fiz.-Tekh. Mat. Nauk 5* (1974) 49-53 (in Russian).

[5] Gadjiev A.D., The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of PP Korovkin, *Soviet Math. Dokl. Vol. 15, No.5* (1974) , 1433-1436

[6] Pai, D.Vand Jain, P.C., Approximation theorems of the Korovkin- type for complex-valued functions. *Indian J., Math. 13* (1971), 123-129.

[7] Gadjiev A.D., *Theorems of Korovkin type*, Mat. Zametki 20 (5) (1976) 781-786; Math. Notes 20 (1976) 995-998 (English transl.)

[8] Ahadov R.A., *On the convergence of sequences of linear operators in a space of functions that are analytic in the disc*, Izv. Akad. Nauk Azerb.
SSR Ser. Fiz-Tekh. Mat. Nauk 2 (1) (1981) 6-71 (in Russian).

[9] Altomare F., Campiti M., *Korovkin-Type Approximation Theory and its applications*, in: de Gruyder Studies in Mathematics, vol. 17, Walter de Gruyter and Co., Berlin, 1994.

[10] Hacisalihoglu H.H:, Gadjiev A.D., *On convergence of sequences of linear positive operators*, Ankara, 1995 (in Turkish)

[11] Ispir N., *Convergence of sequences of k-positive linear operators in subspaces of analytic functions*, Hacet. Bull. Nat. Sci. Eng. Ser. B 28 (1999) 47-53.

[12] Ispir N., Atakut Ç., *On the convergence of a sequence of positive linear operators on the sapce of m-multiple complex sequences*, Hacet. Bull. Nat. Sci. Eng. Ser. B 29 (2000) 47-54.

[13] Gadjiev A. D., Orhan C., *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. 32 (1) (2002) 129-138.

[14] Duman O., *Statistical approximation theorems by k-positive linear operators*, Arch. Math. (Basel) 86 (6)(2006) 569-576.

[15] Ozarslan, M.A., *J-convergence theorems for a class of k-positive linear operators*. Cent. Eur. J. Math. 7 (2),357-362 (2009), 4.

[16] Ozarslan M. A., *I-convergence theorems for a class of k-positive linear operators*, Cent. Eur. J. Math 7 (2) (2009) 357-362.

[17] Gadjiev A.D., Ghorbanalizadeh A.M., *Approximation of analytic functions by the sequences of k-positive linear operators, Journal of Approximation Theory* 162 (2010) 1245-1255.

[18] Gadjiev, A.D., Ghorbanalizadeh, A.M., *On approximation processes in the space of analytic functions.* Centr. Eur. J Math. 8(2), 389-398 (2010).

[19] Gadjiev A. D., *Simaltaneous statistical approximation of analytic functions and their derivatives by k-positive linear operators*, Azerb. J. Math. 1(1) (2011) 57-66.

[20] Gadjiev, A.D., Duman, O., Ghorbanalizadeh, A.M., *Ideal Convergence of k-positive linear operators*. J Funct. Space Appl. Vol. 2012. Article ID 178316 (2012).

[21] Gadjiev A.D., Aliev R.A., *Approximation of analytic functions by k-positive linear operators in the closed domain*, Positivity. 2013 DOI 10. 1007/s 11117-0255-3, 9 p.

[22] Gadjiev A.D., Cetin N., *Approximation of analytic functions by sequences of linear operators*, Filomat 28 (1), 99-106(2014).

[23] Gadjiev, A.D., Aliev, R.A., *Approximation of analytical functions in annulus by linear operators.* Applied mathematics and computations. 252, 438-445 (2015)

[24] Gadjiev, A.D., Aliev, R.A., *Korovkin type theorem for linear k-positive operators in a polydisc of analytical functions.* Mathematica Slovaca 66(5). DOI.10.1515/ms-2016-0213 (2016).

[25] Coşkun T, *Approximation of analytic functions of several variables by linear k-positive operators,* Turk J Math, 41:426-435 (2017).

[26] Freitag, E, Busam, R.: Complex Analysis. pp.9-244. Springer-Verlag, Berlin Heidelberg, New York (2005).

Appendices:

Table 3.1: Numeric values of the approximation to the function $f(z)=\frac{1}{1-z}$ at the point $z=\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$ by the sequences of operators T_n for $n\in\{1,\dots,12\}$.

n	z	$f(z)$	$T_n(f(z))$
10	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$3.572230710+1.953717849\,i$
10^2	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.500561496+1.367602494\,i$
10^3	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.393394575+1.308990959\,i$
10^4	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.382677884+1.303129805\,i$
10^5	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381606214+1.302543690\,i$
10^6	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381499047+1.302485078\,i$
10^7	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381488330+1.302479217\,i$
10^8	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381487258+1.302478631\,i$
10^9	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381487152+1.302478572\,i$
10^{10}	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381487141+1.302478566\,i$
10^{11}	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381487140+1.302478566\,i$
10^{12}	$\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4}i$	$1.190743570+0.6512392828\,i$	$2.381487140+1.302478566\,i$

Table 4.1: Numeric values of the approximation to the function $f(z) = e^z$ at the point $z = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$ by the sequences of operators T_n for $n \in \{1, \dots, 7\}$.

n	z	$f(z)$	$T_n(f(z))$
10	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$3.272897841 + 0.8390818172\, i$
10^2	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.361653025 + 0.4939015277\, i$
10^3	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.336304979 + 0.4930787914\, i$
10^4	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.336037604 + 0.4930778469\, i$
10^5	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.336034914 + 0.4930778459\, i$
10^6	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.336034887 + 0.4930778459\, i$
10^7	$\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} i$	$1.336034887 + 0.4930778459\, i$	$1.336034887 + 0.4930778459\, i$