p-Continuity and p-Homeomorphism in Penta Topological Spaces

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Abstract:
In this paper we introduce the notion of penta topological space and investigate the fundamental concepts in classical topological spaces for penta topological spaces. To do so we define new types of open sets and closed sets in the setting of penta topological space. We also study the idea of p-continuity and p-homeomorphism in penta topological spaces.

Keywords: Penta topological space, p-open set, p-closed set, p-closure, p-cluster point, p-interior, p-exterior, p-continuity, p-homeomorphism.

1. Introduction

In recent years the concept of a single topological space has been extended to bi-topological space (a non-vacuous set X endowed with two topologies τ_1 and τ_2), tri-topological space (a non-vacuous set X endowed with three topologies τ_1, τ_2 and τ_3) and quad-topological space (a non-vacuous set X endowed with four topologies τ_1, τ_2, τ_3 and τ_4). The concept of a bi-topological space was first introduced by Kelly [2]. Later work in the area has been done by Fletcher et al [1], Kim [3], Lane [5], Patty [9], Pervin [10], Reilly [12-14] and others. Tri-topological space was initiated by Kovar [4] and further studied by Priyadharsini and Parvathi [11] and others in different context. Quad-topological space was investigated by

Mukundan [7]. Tapi and Sharma [16] studied the idea of Q-B continuous functions in quad topological spaces. As a natural generalization of these concepts, we introduce a new concept called penta-topological space. A *penta-topological space* $(X,\ \tau)$ is a set X equipped with 5-tuple of topologies $\tau = (\tau_1,\ \tau_2, \tau_3,\ \tau_4, \tau_5)$ - called *penta topology* on X. In this paper we introduce the concept of topological structures with penta topology and define new types of open (closed) sets. We also introduce the notion of p-continuous function and p-homeomorphism in penta- topological spaces.

For details pertaining to classical topological spaces we, however, refer to [8], [15] and [17].

2. Concepts and Results in Penta Topological Spaces

For brevity sake we denote penta topology and penta topological space by p-topology and p-topological space respectively. Symbolically, we write τ_p for p-topology and $(X,\ \tau_p)$ for p-topological space where $\tau_p = (\tau_1, \tau_2, \tau_3,\ \tau_4, \tau_5)$.

Definition 2.1 Let $(X,\ \tau_p)$ be a p-topological space. Elements of τ_i; $i \in \{1, 2, 3, 4, 5\}$ are called τ_i*-open* sets and their relative complements are called τ_i*-closed sets*.

Definition 2.2 Let $(X,\ \tau_p)$ be a p-topological space. A subset A of X is called *penta-open (p-open)* if $A \in \cup\, \tau_i$; $i \in \{1, 2, 3, 4, 5\}$ and its complement is said to be *penta-closed (p-closed).*

Remark 2.3 p-open sets satisfy all the axioms of topology.

Example 2.4 Let $X = \{a, b, c, d\}$ with $\tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{b\}, X\}. \tau_3 = \{\phi, \{c\}, X\}$, $\tau_4 = \{\phi, \{d\}, X\}, \tau_5 = \{\phi, \{a, b\}, X\}$ be a p-topological space. The sets $\phi, X, \{a\}, \{b\}, \{c\}$, $\{d\}$, $\{a,\ b\}$ are p-open sets and ϕ, X, $\{c,\ d\}$, $\{a,\ b,\ c\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d\}$,$\{b,\ c,\ d\}$ are p-closed sets in $(X,\ \tau_p)$.

It is noticed that the sets ϕ and X are both p-open as well as p-closed.

Next we prove some results of fundamental interest concerning p-open sets and p-closed sets.

Theorem 2.5 In $(X,\ \tau_p)$, arbitrary union of p-open sets is p-open.

Proof. Let $\{O_\lambda\!:\ \lambda \in I\}$ be a collection of p-open sets in X. For each $\lambda \in I,\ O_\lambda\ \in \cup\, \tau_i$, $i \in \{1, 2, 3, 4, 5\}$. Therefore $\cup\, O_\lambda \in \cup\, \tau_i, i \in \{1, 2, 3, 4, 5\}$. Hence $\cup\, O_\lambda$ is p-open set.

Theorem 2.6 In $(X,\ \tau_p)$, arbitrary intersection of p-closed sets is p-closed.

Proof. Let $\{C_\lambda :\ \lambda \in I\}$ be a collection of p-closed sets in X. Let $\{O_\lambda\!:\ \lambda \in I\}$ be a collection of p-open sets in X. Put $C_\lambda =\ O_\lambda^c$. Since $\cup\, O_\lambda$ is p-open set, so $(\cup\, O_\lambda)^c$ is p-closed $\Rightarrow \cap\, O_\lambda^c$ is p-closed $\Rightarrow \cap\, C_\lambda$ is p-closed. Hence arbitrary intersection of p-closed sets is closed.

Definition 2.7 Let $(X,\ \tau_p)$ be a p-topological space. Let A be a subset of X. The *p-closure* of A, denoted by p-cl(A) is defined as the intersection of all p-closed sets of X containing A. Thus, if $\{C_\alpha :\ \alpha \in I\}$ is the collection of all p-closed sets in X containing A, then

$$\text{p-cl}(A) = \cap_{\alpha\epsilon I}\ C_\alpha$$

Definition 2.8 A point $x \epsilon X$ is called a *p-closure point* (or a *p-adherent point*) of A if and only if $x \epsilon$ p-cl(A). If p-cl(A) = X, then A is said to be *p-dense* in X.

In Example 2.4, take $A = \{a, b\}$. Then p-cl(A) = $\{a. b\}$ and a, b are the p-closure points of A. In this case A is not p-dense in X since p-cl(A) $\neq X$. However, if A = X, then p-cl(A) = X, and so A is p-dense.

Remarks 2.9

1. $A \subseteq$ p-cl(A).
2. Since intersection of p-closed sets is p-closed, p-cl(A) is a p-closed set
3. p-cl(A) is the smallest p-closed set containing A.

Theorem 2.10 Let (X, τ_p) be a p-topological space and $A \subseteq X$. Then A is p-closed iff A = p-cl(A).
Proof. We have p-cl(A) = $\cap\ \{C\ :\ C \supseteq A,\ C \text{ is p} - \text{closed}\}$. If A is p-closed then A is the member of the above collection and each member contains A. It follows that $\cap\, C = A$. Hence p-cl(A) = A
Conversely, if A = p-cl(A), then A is p-closed, since p-cl(A) is a p-closed set.
Theorem 2.11 Let (X, τ_p) be a p-topological space and $A \subseteq X$. Then (p-cl(A))^c = p-cl(A)^c.
Proof. By taking complement of p-cl(A) and then using the properties of complement, it may easily be verified that (p-cl(A))^c = p-cl(A)^c.

Definition 2.12 Let (X, τ_p) be a p-topological space and $A \subseteq X$. Then A is called *p-neighborhood* of a point $x \in X$ if and only if there exist a p-open set U such that $x \in U \subseteq A$.

Example 2.13 Consider $X, \tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 as defined in Example 2.4. Let $A = \{a, b\}$. Since $\{a\}$ is p-open set and $a \epsilon \{a\} \subseteq A$, therefore A is a p-neighborhood of a.

Definition 2.14 Let (X, τ_p) be a p-topological space and $A \subseteq X$. A point $x \in X$ is called a *p-limit* of A if and only if every p-neighborhood N^p of x contain a point of A other than x. Symbolically, we write, $N^p \cap (A\backslash\{x\}) \neq \phi$
Note that it doesn't make a difference if we restrict the condition to p-open neighborhoods only. The equivalent term such as *p-accumulation point*, *p-cluster point* or *p-derived point* may also be used for p-limit point of a set. The set of all p-limit points of A is called a *p-derived set* of A and is denoted by $p - A^/$ or $p - A^\sim$ or $p - A^d$.

Example 2.15 Let X= {1, 2, 3} with $\tau_1 = \{\phi, \{1\}, X\}$, $\tau_2 = \{\phi, \{1\}, \{1,2\}, X\}$, $\tau_3 = \{\phi, \{1\}, \{1,3\}, X\}$, $\tau_4 = \{\phi, \{1,2\}, X\}$, $\tau_5 = \{\phi, \{2,3\}, X\}$. Take A = {1, 3}, then 2 is the p-limit point of A and $p - A^d = \{2\}$.

Definition 2.16 If (X, τ_p) is a p-topological space and $A \subseteq X$, Then a point $a \in A$ is called a *p-interior point* of A if there exists a p-open set U such that $a \in U \subseteq A$. The set of all p-interior points of A is called the *p-interior* of A and is denoted by p-int*(A).*
In Example 2.15, the points 1 and 3 are clearly the p-interior points of A and p-int(A) = {1, 3}.

Remark 2.17

(i) p-int*(A)* $\subseteq A$. For, if $x \in$ p-int*(A)*, then x is the p-interior point of A i.e. there exists a p-open set U such that $x \in U \subseteq A$. It follows that $x \in A$.

(ii) p-int*(A)* is the union of all p-open sets contained in A and hence the largest p-open set contained in A.

Theorem 2.18 Let (X, τ_p) be a p-topological space and $A \subseteq X$. Then A is p-open iff
A = p-int(A).
Proof. Let A be p-open set in X. Consider the collection $\mathcal{B} = \{O : O \subseteq A, O \text{ is p} - \text{open}\}$. Clearly $A \epsilon \mathcal{B}$ and every member of $\mathcal{B}$ is a subset of A. Therefore $\cup \mathcal{B} = A$ and hence
A = p-int(A).
Conversely, assume that A = p-int(A). Since p-int(A) is p-open set so is A.

Theorem 2.19 Let (X, τ_p) be a p-topological space and $A \subseteq X, B \subseteq$ X. Then
p-int($A \cup B$) $\supseteq$ p-int(A) $\cup$ p-int(B).
Proof. We have p-int(A) $\subseteq A$, p-int(A) is p-open. Also p-int(B) $\subseteq B$, p-int(B) is p-open Then
p-int(A) $\cup$ p-int(B) $\subseteq A \cup B$. Since p-int(A) $\cup$ p-int(B) is a p-open set in $A \cup B$ and p-int($A \cup B$) is the largest p-open set in $A \cup B$, hence p-int($A \cup B$) $\supseteq$ p-int(A) $\cup$ p-int(B).

Definition 2.20 If (X, τ_p) is a p-topological space and $A \subseteq X$, then a point $x \in X$ is called a *p-exterior point* of A if A^c is a p-neighborhood of x. The set of all p-exterior points of A is called the *p-exterior* of A and is denoted by p-ext*(A).*

3. p-Continuity

Definition 3.1 Let (X, τ_p) and $(Y, \tau_p^{/})$ be two p-topological spaces. A function $f: X \longrightarrow Y$ is called a *penta- (or p-) continuous function* if $f^{-1}(V)$ is p-open in X for every p-open set V in Y.

Example 3.2 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{b\}, X\}, \tau_3 = \{\phi, \{c\}, X\}$,
$\tau_4 = \{\phi, \{a, b\}, X\}, \tau_5 = \{\phi, \{a, c\}, X\}$. Let $Y = \{1, 2, 3, 4\}$, $\tau_1^{/} = \{\phi, \{1\}, Y\}$, $\tau_2^{/} = \{\phi, \{1\}, \{1, 3\}, Y\}$, $\tau_3^{/} = \{\phi, \{1\}, \{1, 2\}, Y\}$, $\tau_4^{/} = \{\phi, \{2\}, Y\}$, $\tau_5^{/} = \{\phi, \{3\}, Y\}$.
Define the function $f: X \longrightarrow Y$ by *f(a)* = 1, *f(b)* = 2, *f(c)* = 3. The p-open sets in X are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, X, p-open sets in Y are $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, Y$. Since $f^{-1}(V)$ is p-open in X for every p-open set V in Y, hence f is p-continuous function.

Definition 3.3 Let X and Y be two p-topological spaces. A function $f: X \longrightarrow Y$ is said to be *p- continuous at a point* $x \epsilon X$ if for every p-open set V containing $f(x)$ in Y, there exist a p-open set U containing x such that $f(U) \subseteq V$.

Theorem 3.4 A function $f : X \longrightarrow Y$ is p-continuous iff f is p-continuous at each point of X.
Proof. Let $f : X \longrightarrow Y$ be a p-continuous function. Let V be a p-open set containing $f(x)$ for any x in X. Since f is a p-continuous function so $f^{-1}(V)$ is p-open set containing x. Let $f^{-1}(V) = U$. Then $f(U) = V$ is a p-open set implies that there exists a p-open set U containing x. Hence f is p-continuous at x. Since x was chosen arbitrary, so f is p-continuous at each point of X.
Conversely, suppose that f is p-continuous at each point of X. Let V be a p-open set of Y. If $f^{-1}(V) = \phi$ then it is p-open. Consider any x in $f^{-1}(V)$. Since f is p-continuous at x, hence there exists a p-open set U_x containing x and $f(U_x) \subseteq V$. Let $U = \cup\{U_x : x \epsilon f^{-1}(V)\}$. We prove that $U = f^{-1}(V)$. For $x \epsilon f^{-1}(V) \Rightarrow U_x \subseteq U \Rightarrow x \epsilon U. If\ x \in U \Rightarrow x \epsilon U_x$ for some $x \Rightarrow x \epsilon V \Rightarrow x \epsilon f^{-1}(V). Hence\ U = f^{-1}(V)$. Each U_x is p-open, so U is p-open. Therefore $f^{-1}(V)$ is p-open in X. Consequently f is p-continuous.

Theorem 3.5 Let (X, τ_p) and $(Y, \tau_p^{/})$ be two p-topological spaces. A function $f : X \longrightarrow Y$ is p- continuous iff $f^{-1}(V)$ is p-closed in X whenever V is p-closed in Y.
Proof. Let $f : X \longrightarrow Y$ be a p-continuous function. Let V be a p-closed set in Y. Then $f^{-1}(V)$ is p-open in $X \Rightarrow f^{-1}(V^c)$ is $\mathrm{p-closed}$ in $X \Rightarrow [f^{-1}(V)]^c$ is p-closed in $X \Rightarrow f^{-1}(V)$ is p-closed in X. Hence $f^{-1}(V)$ is p-closed in X whenever V is p-closed in Y.
Conversely, suppose $f^{-1}(V)$ is p-closed in X whenever V is p-closed in Y. V is a p-open set in $Y \Rightarrow V^c$ is p-closed in $Y \Rightarrow f^{-1}(V^c)$ is p-closed in $X \Rightarrow [f^{-1}(V)]^c$ is p-closed in $X \Rightarrow f^{-1}(V)$ is p- open in X. Hence f is p-continuous.

Theorem 3.6 A $f : X \longrightarrow Y$ is p-continuous iff f[p-cl(A)] $\subseteq$ p-cl[$f(A)$] for all $A \subseteq X$.
Proof. Suppose $f : X \longrightarrow Y$ is a p-continuous function. Since p-cl[$f(A)$] is p-closed in Y, then by Theorem 3.5 $f^{-1}(p-\mathrm{cl}[f(A)])$ is p-closed in X,
Note that p-cl$[f^{-1}(p-\mathrm{cl}(f(A)))] = f^{-1}[p-\mathrm{cl}(f(A))]$. (*)
Now $f(A) \subseteq p - \mathrm{cl}[f(A)]$, $A \subseteq f[f^{-1}(A)] \subseteq f^{-1}[p - \mathrm{cl}(f(A))]$. Then
p-cl(A)$\subseteq p - \mathrm{cl}[f^{-1}(p - \mathrm{cl}(f(A)))] = f^{-1}(p - \mathrm{cl}[f(A)])$ by (*) which yields
f[p-cl(A)] $\subseteq p - cl[f(A)]$.
Conversely, let f[p-cl(A)] $\subseteq$ p-cl[$f(A)$] for all $A \subseteq X$. Let F be a p-closed set in Y, so that p-cl(F) = F. Now $f^{-1}(F) \subseteq X$, so by hypothesis,
f[p-cl(f^{-1} (F))] $\subseteq p - \mathrm{cl}[f(f^{-1}(F))] \subseteq p - \mathrm{cl}(F) = F$. Therefore p-cl($f^{-1}$ (F)) $\subseteq f^{-1}$ (F).
But $f^{-1}(F) \subseteq$ p-cl(f^{-1} (F)) always. Hence p-cl$[f^{-1}(F)] = f^{-1}(F)$ and so $f^{-1}(F)$ is p-closed in X. Hence by Theorem 3.5, f is p- continuous.

4. p-Homeomorphism

Definition 4.1 Let $(X,\ \tau_p)$ and $(Y,\ \tau_p^{/})$ be two p-topological spaces. A function $f\colon X \longrightarrow Y$ is called *p-open (resp. p-closed) map* if $f(V)$ is p-open (resp. p-closed) in Y for every p-open (resp. p-closed) set V in X.

Example 4.2 In Example 3.2 f is p-open (resp. p-closed) map.

Theorem 4.3 Let $(X,\ \tau_p)$ and $(Y,\ \tau_p^{/})$ be two p-topological spaces. A mapping $f\colon X \longrightarrow Y$ is p-continuous if and only if $f^{-1}: Y \longrightarrow X$ is p-open map.
Proof. Trivial.

Definition 4.4 Let $(X,\ \tau_p)$ and $(Y,\ \tau_p^{/})$ be two p-topological spaces. A mapping $f\colon X \longrightarrow Y$ is called a *Penta*-(or *p-) homeomorphism*, if

a, f is a bijection
b. f is p-continuous
c. f^{-1} is p-continuous

The p-topological spaces X, Y are said to be *p-homeomorphic*, written as $X \overset{p}{\cong} Y$, if there is a p-homeomorphism $f\colon X \longrightarrow Y$.

Example 4.5 The function f defined in Example 3.2 is clearly a p-homeomorphism and $X \overset{p}{\cong} Y$.

Remark 4.6 Observe that p-homeomorphism is an equivalence relation in the class of all p-topological spaces, since

(a) $X \overset{p}{\cong} X$
(b) $X \overset{p}{\cong} Y \Rightarrow Y \overset{p}{\cong} X$
(c) $X \overset{p}{\cong} Y, Y \overset{p}{\cong} Z \Rightarrow X \overset{p}{\cong} Z$

Remark 4.7 p-homeomorphism is both p-open and p-closed map.

5. Conclusion

In this paper we introduced the notion of penta topological space and defined new types of open and closed sets that is p-open set and p-closed set in penta topological space. Some properties of p-open sets and p-closed sets are studied. The idea of p-continuity and p-homeomorphism is also introduced in penta topological spaces. Further work is needed to carry over other concepts such as separation axioms, compactness, connectedness etc. of classical topological spaces to penta topological spaces.

References

[1] Fletcher, P., Hoyle, H. B., III, and Patty, C. W. (1969), The comparison of topologies. Duke Math. J. 56: 525-551.

[2] Kelly, J. C. (1963). Bitopological spaces. Proc. London Math. Soc., 13(3), 71—89.

[3] Kim, Y. W. (1968), Pairwise compactness. Publ. Math. Debrecen, 15: 87-90.

[4] Kovar , M.(2000), On 3-Topological Version Of Theta-Regularity, Internat. J. Math. Math. Sci., Vol. 23,No.6, 393–398.

[5] Lane, E. P.(1965), Bitopological spaces. Unpublished Ph.D. thesis. Lafayette, Ind., Library, Purdue University.

[6] Lellis,T. M. (1991), Generalization of pairwise α-continuous functions, Pure and Applied Mathematicka Sciences, Vol.28, 55-63.

[7] Mukundan, D. V. (2013), Introduction to Quad topological spaces (4-tuple topology), International Journal of Scientific & Engineering Research, Volume 4, Issue 7, 2483-2485

[8] Munkres J.R.(2000), Topology (A first course) Prentice Hall Inc.

[9] Patty, C. W. (1967), Bitopological spaces. Duke Math. J. 54: 587-592.

[10] Pervin, W. J.(1967), Connectedness in bitopological spaces. Nederl. Akad. Wetensch Proc. Ser. A, 70: 569-572.

[11] Priyadharsini P. and Parvathi A. (2017), Tri-b-Continuous Function in Tri Topological Spaces, International Journal of Mathematics And its Applications, Volume 5, Issue 4F, 959-962.

[12] Reilly, Ivan L.(1970), Quasi-gauges, quasi-uniformities and bitopological spaces. Unpublished Ph.D. thesis, Urbana-Champaign, 111., Library, University of Illinois.

[13] Reilly, I. L. (1972). On bitopological separation properties. Nanta Math., (2), 14—25.

[14] Reilly, I. L. (1973), Zero dimensional bitopological spaces. Indag. Math., (35) 127–131.

[15] Simmons G.F. (1963), Introduction to Topology and Modern Analysis, Mc Graw-Hill Book Company.

[16] Tapi, U.D., Sharma, R. (2015), Q- B Continuous Function In Quad Topological Spaces, International Journal of Mathematics And Computer Research, Volume 3, issue 6, 1011-1017.

[17] Willard, S. (1970), General Topology, Addison Wesley, N.Y.