FROM DYNAMICAL POLYSYSTEMS TO CONTROL SYSTEMS AND BACK

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Abstract. This work shows a generalization of control systems (as defined by E. Sontag) via dynamical polysystems and establishes the equivalence of the two notions under a certain lipschitz condition on the function defining the dynamics.

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1. Introduction

The notion of *dynamical polysystem* appeared in the 1970's, being introduced by C. Lobry, [2]. It had the following meaning: a dynamical polysystem on a manifold M is a family

$$\mathcal{F}_{pc} = \{\mathcal{F}(\cdot, u) : u \in \mathcal{U}_{pc}\}$$

of smooth vector fields depending on a piecewise constant parameter u, called *input*. A similar meaning was given to dynamical polysystems in the work of J. Tsinias and N. Kalouptsidis, [3].

In this paper, a dynamical polysystem is regarded in a slightly more general way, as a family of continuous dynamical systems, all defined on the same metric space X, not necessarily by means of differential equations. The analogy between dynamical polysystems and control systems with piecewise constant inputs is quite natural. Intuitively, a motion in a dynamical polysystem means starting at a point $x \in X$, traveling for a time t_1 according to a dynamical system Φ_1, then switching to another dynamical system Φ_2 and traveling for a time t_2, and so forth.

The main goal of this work is to create a bridge between continuous dynamical systems and control systems, by means of dynamical polysystems. In future investigations the author will consider adapting results of stability in dynamical polysystems to control systems. The first part of this paper shows how a generalized control system can arise from a family of continuous dynamical systems on a metric space X, provided the family of dynamical systems satisfies a condition of smooth dependence on indices. The second part is dedicated to proving that, indeed, the theory allows a consistent generalization of control systems that are usually defined by differential equations with parameters, under the lipschitz condition

$$|f(x, m) - f(y, m')| \leq \alpha|x - y| + \beta d(m, m').$$

on the function f defining the dynamics.

2. Definitions

Consider a family $\mathcal{F}$ of continuous dynamical systems, all defined on a metric space X. For any $\phi \in \mathcal{F}$ and $t \in \mathbb{R}$, $\phi_t(x) = \phi(t, x)$ defines a homeomorphism ϕ_t on X, having inverse ϕ_{-t}.

Definition 1. *Let $\mathcal{G}$ be the subgroup of $(\mathbb{R} \times Homeo(X), (+, \circ))$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \in \mathbb{R}\}$. The pair $(\mathcal{G}, X)$ is called a* ***dynamical polysystem*** *on X. The* ***accessibility semigroup*** *of $\mathcal{G}$, denoted by $\mathcal{S}$, is the subsemigroup of $\mathcal{G}$ generated by $\{(t, \phi_t) : \phi \in \mathcal{F}, t \geq 0\}$. The pair $(\mathcal{S}, X)$ is called the* ***accessibility polysystem*** *on X generated by $\mathcal{F}$.*

Remark 1. *An element of $\mathcal{G}$ has form*

$$g = (t, h) = (t_1 + t_2 + ... + t_k, \phi^1_{t_1} \circ \phi^2_{t_2} \circ ... \circ \phi^k_{t_k}), \tag{1}$$

with $t_i \in \mathbb{R}$ and $\phi^i \in \mathcal{F}$, for $0 \leq i \leq k$.

The group $\mathcal{G}$ acts on X by $(t,h).x = h(x)$. Moreover, any subsemigroup of $\mathcal{G}$ acts on X by restricting this action. A **subsystem** of $(\mathcal{G}, X)$ can be defined in a natural way, by restricting $\mathcal{F}$ to a subset.

The polysystem $(\mathcal{G}, X)$ can be considered (and, in fact, is) a $\mathcal{G}$-dynamical system. In what follows, though, notions related to dynamical systems in general may be defined or approached differently, given the concern for regarding polysystems in close connection with continuous-time dynamical systems.

We note a similarity between dynamical polysystems and control systems. Our first objective is to show that a dynamical polysystem gives rise to a control system in a natural way when regulated functions are considered as controls (sections 3 and 4). We will then show that a control system can be viewed as a dynamical polysystem provided a lipschitz condition is satisfied (section 5).

In what follows, let X be a complete metric space, M a separable metric space, and $\Phi : \mathbb{R} \times X \times M \to X$ a continuous function with the property that every $\Phi_m : \mathbb{R} \times X \to X$ defined by $\Phi_m(t,x) := \Phi(t,x,m)$ is a continuous dynamical system on X.

We will now focus on constructing a control system, in the sense of Sontag [1], with a transition function Ψ naturally arising from Φ, under certain hypotheses on Φ.

Definition 2. *Let $u : [a,b] \to M$. We say that u is a **regulated function** if it is the uniform limit of a sequence of piecewise constant functions.*

It is shown [4, pp. 60], that a necessary and sufficient condition for a function to be regulated is that it has a limit from the left and a limit from the right at every point in its domain (at end points, just the applicable one-sided limit).

Definition 3. *Let $u : [0,T] \to M$ and $v : [0,S] \to M$. The **concatenation** $u * v : [0, T+S] \to M$ (also called the **Myhill product**) of u and v is defined by*

$$u * v(t) := \begin{cases} u(t) & , \text{ if } 0 \le t < T\ ; \\ v(t-T) & , \text{ if } T \le t \le T+S. \end{cases}$$

Remark 2. *The operation of concatenation preserves the uniform limit. That is, if $u_n \xrightarrow{u} u$ and $v_n \xrightarrow{u} v$ then $u_n * v_n \xrightarrow{u} u * v$.*

3. Piecewise constant controls

Definition 4. *Let $T > 0$ and consider $u : [0,T] \to M$, a piecewise constant function defined by a finite partition $0 = t_0 < t_1 < t_2 < \ldots < t_k = T$ of the interval $[0,T]$, and elements $m_1, m_2, \ldots, m_k$ of M with $u(t) = m_i$ whenever $t \in (t_{i-1}, t_i)$, for $i \in \{1, 2, \ldots, k\}$. Note that the values of u at the points $t_0, t_1, \ldots, t_k$ are ignored, for they are unimportant when defining the transition function.*

For $x \in X$ and u as above, define the sequence:
$x_1 := \Phi(t_1, x, m_1)$,
$x_2 := \Phi(t_2 - t_1, x_1, m_2)$,
. . . .
$x_k := \Phi(t_k - t_{k-1}, x_{k-1}, m_k)$,
and set $\Psi(T, x, u) := x_k$.

Remark 3. *The function Ψ is well defined, that is $\Psi(T, x, u)$ is independent of the representation of the piecewise constant function u.*

Proof. Let u have an arbitrary representation as above. We also consider the unique representation u^* given by maximal intervals of constancy. Express the latter using $0 = s_0 < s_1 < s_2 < ... < s_l = T$ so that $u^*(t) = m'_i$, when $t \in (s_{i-1}, s_i)$ and $m'_i \neq m'_{i+1}$ for all i. Now, construct the sequence:

$x'_1 := \Phi(s_1, x, m'_1)$,
$x'_2 := \Phi(s_2 - s_1, x'_1, m'_2)$,
. . . .
$x'_l := \Phi(s_l - s_{l-1}, x'_{l-1}, m'_l)$.

Then there exist indices $k_1 < k_2 < ... < k_l$ for which

$$t_{k_1} = s_1, t_{k_2} = s_2, ..., t_{k_l} = s_l.$$

Using induction, we can easily prove that $x_{k_i} = x'_i$ for all $i \in \{1, 2, ..., l\}$. It suffices to show it for $i = 1$, since the same verification is used for the inductive step. Noting that $m_1 = m_2 = ... = m_{k_1} = m'_1$, we have:

$$\begin{aligned} x'_1 &= \Phi(s_1, x, m'_1) = \\ &= \Phi(s_1 - t_1, \Phi(t_1, x, m'_1), m'_1) = \Phi(s_1 - t_1, \Phi(t_1, x, m_1), m'_1) = \\ &= \Phi(s_1 - t_1, x_1, m'_1) = \Phi(s_1 - t_2, \Phi(t_2 - t_1, x_1, m'_1), m'_1) = \\ &= \Phi(s_1 - t_2, \Phi(t_2 - t_1, x_1, m_2), m'_1) = \Phi(s_1 - t_2, x_2, m'_1) = \\ &... \\ &= x_{k_1}. \end{aligned}$$

The only fact used in this sequence of equalities is the semigroup property of the dynamical system $\Phi_{m'_1}$. □

Proposition 1. *The function Ψ defined above satisfies the semigroup property for piecewise constant controls:*

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v), \tag{2}$$

where $u : [0, T] \to M$ and $v : [0, S] \to M$ are piecewise constant functions.

Proof. Note that if $u(T) \neq v(0)$ then the property holds by the mere construction of $\Psi(T + S, x, u * v)$.

Without loss of generality, we can then reduce the problem to the case when u and v are both constant functions and, moreover, they are defined by the same constant, say m. The property becomes, in this case,

$$\Phi(T+S,x,m)=\Phi(S,\Phi(T,x,m),m).$$

This is true indeed, by the semigroup property of the dynamical system Φ_m. □

4. Regulated controls

In order to extend the definition of the function Ψ, we will assume that the given polysystem satisfies the following hypothesis. Lemma 1 will then justify the technicalities involved.

H 1. *For every $T>0$ there exists a continuous function $K:[0,T]\to[1,\infty)$ such that*

$$K(t_1)K(t_2)\leq K(t_1+t_2) \text{ for every } t_1 \text{ and } t_2 \tag{3}$$

and

for every $\epsilon>0$ there exists a $\delta>0$ such that $x_0, y_0\in X$, $t\in[0,T]$, and $d(m_0,n_0)<\delta$ imply

$$d(\Phi(t,x_0,m_0),\Phi(t,y_0,n_0))\leq K(t)d(x_0,y_0)+\epsilon\int_0^t K(s)ds. \tag{4}$$

Consider $x\in X, T>0, u:[0,T]\to M$, a regulated function, and $\{u_n\}_n$ a sequence of piecewise constant functions on $[0,T]$, converging uniformly to u.

Fix a positive integer n and assume that u_n is defined by a partition $0=t_0^n<t_1^n<t_2^n<...<t_{k_n}^n=T$ of the interval $[0,T]$ and elements $m_1^n, m_2^n, ..., m_{k_n}^n$ of M with $u_n(t)=m_i^n$ whenever $t\in(t_{i-1}^n,t_i^n)$, for $i\in\{1,2,...,k_n\}$.

Following the procedure described in Definition 4, construct the sequence:

$x_1^n := \Phi(t_1^n, x, m_1^n)$,
$x_2^n := \Phi(t_2^n - t_1^n, x_1^n, m_2^n)$,
. . . .
$x_{k_n}^n := \Phi(t_{k_n}^n - t_{k_n-1}^n, x_{k_n-1}^n, m_{k_n}^n)$.

Using Definition 4, we can rename the last element of this sequence:

$$x_n := x_{k_n}^n = \Psi(T,x,u_n). \tag{5}$$

Lemma 1. *Assume that Φ satisfies hypothesis 1. Given $T>0, x\in X, u$, and $\{u_n\}_n$ as above, the sequence $\{x_n\}_n$, defined by equation (5), is Cauchy (hence convergent, by the completeness of X).*

Proof. Let K be a function as in hypothesis 1 and set $A := \int_0^T K(s)ds$.

Let $\epsilon > 0$. Find $\delta > 0$ such that equation (4) is satisfied for $\frac{\epsilon}{A}$, that is:

$$d(\Phi(t, x_0, m_0), \Phi(t, y_0, n_0)) \leq K(t)d(x_0, y_0) + \frac{\epsilon}{A} \int_0^t K(s)ds,$$

whenever $d(m_0, n_0) < \delta, x_0, y_0 \in X, t \in [0, T]$.

Since $u_n \xrightarrow{n} u$ uniformly, there exists $N \in \mathbb{N}$ such that $p, q \geq N$ implies $d(u_p(t), u_q(t)) < \delta$, for all $t \in [0, T]$.

Fix $p, q \geq N$.

As in the discussion preceding equation (5), u_p and u_q are defined using partitions of the interval $[0, T]$: $0 = t_0^p < t_1^p < t_2^p < ... < t_{k_p}^p = T$ and $0 = t_0^q < t_1^q < t_2^q < ... < t_{k_q}^q = T$, respectively. Letting $0 = s_1 < s_2 < ... < s_k = T$ be the common refinement of the two partitions, we see that $u_p(t) = m_i^p$ and $u_q(t) = m_i^q$, whenever $t \in (s_{i-1}, s_i)$, for all $i \in \{1, 2, ..., k\}$. So,

$$d(m_i^p, m_i^q) < \delta, \text{ for all } i \in \{1, 2, ..., k\}. \tag{6}$$

Now, we prove by induction on j that

$$d(x_j^p, x_j^q) \leq \frac{\epsilon}{A} \int_0^{s_j} K(s)ds, \text{ for all } j \in \{1, 2, ..., k\}. \tag{7}$$

For $j = 1$ we have:

$$d(x_1^p, x_1^q) = d(\Phi(s_1, x, m_1^p), \Phi(s_1, x, m_1^q) \leq \frac{\epsilon}{A} \int_0^{s_1} K(s)ds,$$

by properties (4) and (6).

Now we prove that if inequality (7) holds for j, then it holds for $j + 1$.

$$\begin{aligned} d(x_{j+1}^p, x_{j+1}^q) &= d(\Phi(s_{j+1} - s_j, x_j^p, m_{j+1}^p), \Phi(s_{j+1} - s_j, x_j^q, m_{j+1}^q)) \\ &\leq K(s_{j+1} - s_j)d(x_j^p, x_j^q) + \frac{\epsilon}{A} \int_0^{s_{j+1}-s_j} K(s)ds, \end{aligned}$$

by property (4). Using the inductive hypothesis, we then have:

$$\begin{aligned} d(x_{j+1}^p, x_{j+1}^q) &\leq K(s_{j+1} - s_j)\frac{\epsilon}{A} \int_0^{s_j} K(s)ds + \frac{\epsilon}{A} \int_0^{s_{j+1}-s_j} K(s)ds \\ &= \frac{\epsilon}{A} \left[\int_0^{s_j} K(s_{j+1} - s_j)K(s)ds + \int_0^{s_{j+1}-s_j} K(s)ds \right]. \end{aligned}$$

Now, using property (3) of $K(\cdot)$, we may write:

$$\begin{aligned}
d(x^p_{j+1}, x^q_{j+1}) &\leq \frac{\epsilon}{A}\left[\int_0^{s_j} K(s_{j+1} - s_j + s)ds + \int_0^{s_{j+1}-s_j} K(s)ds\right] \\
&= \frac{\epsilon}{A}\left[\int_{s_{j+1}-s_j}^{s_{j+1}} K(t)dt + \int_0^{s_{j+1}-s_j} K(s)ds\right] \\
&= \frac{\epsilon}{A}\int_0^{s_{j+1}} K(s)ds.
\end{aligned}$$

It follows that $d(x_p, x_q) \leq \frac{\epsilon}{A}\int_0^T K(s)ds = \epsilon$, proving that $\{x_n\}_n$ is a Cauchy sequence. □

We summarize the results of Lemma 1 in the following corollary.

Corollary 1. *Under hypothesis 1, if $T > 0, x \in X, u : [0, T] \to M$ is a regulated function, and $\{u_n\}_n$ is a sequence of piecewise constant functions uniformly converging to u, then*

$$\lim_{n\to\infty} \Psi(T, x, u_n) \text{ exists in } X.$$

Lemma 2. *If Φ satisfies hypothesis 1, $T > 0, x \in X, u : [0, T] \to M$ is a regulated function, and $\{u_n\}_n, \{v_n\}_n$ are two sequences of piecewise constant functions, both uniformly converging to u, then*

$$\lim_{n\to\infty} \Psi(T, x, u_n) = \lim_{n\to\infty} \Psi(T, x, v_n).$$

Proof. Let $\{w_n\}_n$ be the sequence constructed by:

$$w_n := \begin{cases} u_k & \text{, if } n = 2k+1 \;; \\ v_k & \text{, if } n = 2k. \end{cases}$$

It is not difficult to show that $w_n \xrightarrow{n} u$ uniformly. To see this, let $\epsilon > 0$ and let $N_1, N_2 \in \mathbb{N}$ be such that

$$n \geq N_1 \text{ implies } d(u_n(t), u(t)) < \epsilon, \text{ for all } t \in [0, T]$$

and

$$n \geq N_2 \text{ implies } d(v_n(t), u(t)) < \epsilon, \text{ for all } t \in [0, T].$$

Set $N := 2 \cdot max(N_1, N_2)$. If $n \geq N$ then $d(w_n(t), u(t)) < \epsilon$, for all $t \in [0, T]$. Therefore,

$$\lim_{k\to\infty} \Psi(T, x, u_k) = \lim_{k\to\infty} \Psi(T, x, w_{2k+1}) = \lim_{k\to\infty} \Psi(T, x, w_{2k}) = \lim_{k\to\infty} \Psi(T, x, v_k).$$

□

Definition 5. *If Φ satisfies hypothesis 1, $T > 0, x \in X$, and $u : [0, T] \to M$ is a regulated function , define*

$$\Psi(T, x, u) := \lim_{n\to\infty} \Psi(T, x, u_n),$$

where $\{u_n\}_n$ *is any sequence of piecewise constant functions uniformly converging to* u*.*

Note that the consistency of this definition is assured by Lemma 2.

Lemma 3. *If* $u : [0,T] \to M$ *is a piecewise constant function then the map*

$$x \mapsto \Psi(T, x, u)$$

is continuous on X*.*

Proof. Let u have the unique representation given by maximal intervals of constancy and let k be the number of these intervals. We will prove the result inductively by k.

If $k = 1$ then u is constant on $[0,T]$, say $u \equiv m \in M$. Then $\Psi(T, x, u) = \Phi(T, x, m)$ and the result follows from the continuity of Φ in the second argument.

Now suppose that the result is true for any piecewise constant function with k maximal intervals of constancy and let u have $k+1$ maximal intervals of constancy. Specifically, assume that $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = T$ are such that $u|_{(t_{i-1},t_i)} \equiv m_i$ for $i \in \{1, 2, \ldots, k+1\}$. Then

$$\Psi(T, x, u) = \Phi(t_{k+1} - t_k, \Psi(t_k, x, u|_{[0,t_k]}), m_{k+1})$$

and the result follows from the inductive hypothesis combined with the continuity of Φ in the second variable.

□

Proposition 2. *The function* Ψ *satisfies the semigroup property for regulated controls:*

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v),$$

where $u : [0,T] \to M$ *and* $v : [0,S] \to M$ *are regulated functions.*

Proof. Let $u : [0,T] \to M$ and $v : [0,S] \to M$ be regulated functions. Consider two sequences of piecewise constant functions, $\{u_n\}_n$ and $\{v_k\}_k$, with $u_n \overset{u}{\to} u$ and $v_k \overset{u}{\to} v$. Then, fixing a $k \in \mathbb{N}$, note that $u_n * v_k \overset{n}{\to} u * v_k$ uniformly, by Remark 2. We have:

$$\Psi(T + S, x, u * v_k) = \lim_{n\to\infty} \Psi(T + S, x, u_n * v_k),$$

by the definition of $\Psi(T + S, x, u * v_k)$. Then, using Proposition 1, we may continue and write

$$\lim_{n\to\infty} \Psi(T + S, x, u_n * v_k) = \lim_{n\to\infty} \Psi(S, \Psi(T, x, u_n), v_k).$$

Using Lemma 3, we see that

$$\lim_{n\to\infty} \Psi(S, \Psi(T, x, u_n), v_k) = \Psi(S, \lim_{n\to\infty} \Psi(T, x, u_n), v_k) = \Psi(S, \Psi(T, x, u), v_k).$$

Thus,

$$\Psi(T + S, x, u * v_k) = \Psi(S, \Psi(T, x, u), v_k), \text{ for every } k.$$

Now, taking the limit as k tends to infinity of both sides, we conclude that

$$\Psi(T + S, x, u * v) = \Psi(S, \Psi(T, x, u), v).$$

Note, again, that for this last step we only use the definition of Ψ. $\square$

We thus showed how a dynamical polysystem (Φ) gives rise to a control system with regulated functions as controls (Ψ).

5. Control systems via dynamical polysystems

In this section we show how a control system naturally induces a dynamical polysystem and how hypothesis 1 becomes more "friendly" when a lipschitz condition is considered.

Let $X = \mathbb{R}^n$, let M be a metric space , and let $f : X \times M \to \mathbb{R}^n$ be a function satisfying the conditions of being a *right hand side*, (*rhs* [1]), that is:

$$f(\cdot, m) \text{ is of class } C^1 \text{ for each fixed } m$$

and

$$f \text{ and } \frac{\partial f}{\partial x} \text{ are continuous on } X \times M.$$

This *rhs* gives rise to a time-invariant continuous-time control system, whose transition function $\Gamma(t, x_0, u)$ is defined to be the unique maximal solution of the initial value problem

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0. \end{cases} \tag{8}$$

For details on this control system one may consult [1, pp. 44].

We would like to define a family of dynamical systems on X, using the above system and constant controls. The control system itself can be defined on arbitrary open subsets X of $\mathbb{R}^n$; but in order to avoid additional constraints, for the family of dynamical systems constructed below, we want to assume that X is the whole of $\mathbb{R}^n$. Then the existence of global solutions for the initial value problem (8) is required. And this existence is guaranteed by the following result, a direct consequence of Proposition C.3.8 in [1].

Proposition 3. *Assume that the rhs f satisfies the property that for every $m \in M$ there exists a constant $\alpha > 0$ so that*

$$|f(x, m) - f(y, m)| \leq \alpha |x - y|, \tag{9}$$

for all x and y in $X = \mathbb{R}^n$. Then the initial value problem (8) admits global solutions, for every constant control $u \equiv m$.

Now, Proposition 3 allows us to construct a family of dynamical systems on $X = \mathbb{R}^n$. For $t > 0, x \in X,$ and $m \in M$, set

$$\phi(t, x, m) := \Gamma(t, x, u), \text{ where } u : [0, t] \to M, u \equiv m. \tag{10}$$

In other words, $\phi(t, x, m)$ is defined to be the unique solution of the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}, m) \\ \mathbf{x}(0) = x. \end{cases} \tag{11}$$

The continuity of ϕ involves smooth dependence on initial conditions and parameters. The following result is needed.

Theorem 1. *(Brauer-Nohel,* [5, pp. 331]*) Let $g, h : D \to \mathbb{R}^n$ be bounded functions of class $\mathbf{C^1}$ and let $K > 0$ be an upper bound for $|g'(x)|$ as well as for $|h'(x)|$ on D. Let ω and τ be solutions of $\dot{x} = g(x)$ and $\dot{x} = h(x)$, respectively, with $\omega(0) = x_0$ and $\tau(0) = y_0$, existing on a common interval $[0, T]$. Suppose $|g(x) - h(x)| \le \eta$, for all $x \in D$. Then ω and τ satisfy the estimate*

$$|\omega(t) - \tau(t)| \le |x_0 - y_0| e^{Kt} + \eta T e^{Kt},$$

for all $t \in [0, T]$.

Proposition 4. *The map ϕ is continuous on $[0, \infty) \times X \times M$.*

Proof. Fix $t_0 > 0, x_0 \in X$ and $m_0 \in M$. We will show that ϕ is continuous at (t_0, x_0, m_0).

Consider $\epsilon > 0$.

If ξ is the (global) solution of the initial value problem (11), with $x = x_0$ and $m = m_0$, by the continuity of ξ at t_0 it follows that there exists a $\nu > 0$ such that, for every $t \in (t_0 - \nu, t_0 + \nu)$,

$$|\xi(t) - \xi(t_0)| \le \frac{\epsilon}{2},$$

or, equivalently,

$$|\phi(t, x_0, m_0) - \phi(t_0, x_0, m_0)| \le \frac{\epsilon}{2}.$$

By the continuity of $f(\cdot, \cdot)$, there exist neighborhoods A of x_0 and B of m_0 and a positive number K such that $|f(x, m)| \le K$ for all $x \in A$ and $m \in B$.

Let $\eta := \epsilon \frac{e^{-K(t_0+\nu)}}{4(t_0+\nu)}$.

Let $\delta > 0$ be such that $\delta < \eta(t_0 + \nu)$, $B(x_0, \delta) \subset A$, $B(m_0, \delta) \subset B$, and $|f(x, m) - f(x, m_0)| \le \eta$, whenever $x \in B(x_0, \delta)$ and $m \in B(m_0, \delta)$. We may also assume that $\delta < \nu$.

Let $t \in B(t_0, \delta)$, $x \in B(x_0, \delta)$, and $m \in B(m_0, \delta)$.

Now, set $g(\cdot) := f(\cdot, m), h(\cdot) := f(\cdot, m_0), D := B(x_0, \delta)$, and $T := t$, and apply Theorem 1 to obtain

$$|\phi(t, x, m) - \phi(t, x_0, m_0)| \le |x - x_0| e^{Kt} + \eta t e^{Kt}.$$

Since $t < t_0 + \nu$ and $|x - x_0| \le \delta < \eta(t_0 + \nu)$, we have

$$|\phi(t, x, m) - \phi(t, x_0, m_0)| \le e^{K(t_0+\nu)}[\eta(t_0+\nu) + \eta(t_0+\nu)] \le 2e^{K(t_0+\nu)}\eta(t_0+\nu) = \frac{\epsilon}{2}.$$

Using the triangle inequality, we conclude that

$$|\phi(t, x, m) - \phi(t_0, x_0, m_0)| \le |\phi(t, x, m) - \phi(t, x_0, m_0)| + |\phi(t, x_0, m_0) - \phi(t_0, x_0, m_0)|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

The following result, known as Gronwall's Lemma, is needed for the main result in this section. For a detailed proof see [6, pp. 179].

Lemma 4. *Let $z : [0, T] \to \mathbb{R}^n$ satisfy*

$$|\dot{z}(t)| \leq \gamma |z(t)| + c, \text{ for all } t \in [0, T],$$

where γ and c are non-negative constants.
Then, for every $t \in [0, T]$, we have

$$|z(t) - z(0)| \leq (e^{\gamma t} - 1)|z(0)| + c \int_0^t e^{\gamma(t-s)} ds.$$

Proposition 5. *Let the rhs f satisfy the lipschitz condition that there exist positive constants α and β such that for every $x, y \in X, m, m' \in M$,*

$$|f(x, m) - f(y, m')| \leq \alpha |x - y| + \beta d(m, m'). \tag{12}$$

Then ϕ satisfies Hypothesis 1.

Proof. Note, first, that condition (9) is automatically satisfied, so the definition of ϕ makes sense.

Let $T > 0$ and define $K : [0, T] \to [1, \infty)$ by $K(t) := e^{\alpha t}$. Clearly,

$$K(t_1)K(t_2) = K(t_1 + t_2), \text{ for all } t_1, t_2.$$

Now, let $\epsilon > 0$ and set $\delta := \epsilon/\beta$.

Fix $x_0, y_0 \in X$, $t \in [0, T]$, and $m, m' \in M$ with $d(m, m') \leq \delta$. For simplicity, denote $x(t) := \phi(t, x_0, m)$ and $y(t) := \phi(t, y_0, m')$ and set $z(t) := x(t) - y(t)$. Then $\dot{x}(t) = f(x(t), m)$, $\dot{y}(t) = f(y(t), m)$, $x(0) = x_0$, and $y(0) = y_0$. So, we may write

$$|\dot{z}(t)| = |\dot{x}(t) - \dot{y}(t)| = |f(x(t), m) - f(y(t), m')|,$$

and then, using (12),

$$|\dot{z}(t)| \leq \alpha |x(t) - y(t)| + \beta d(m, m') \leq \alpha |z(t)| + \epsilon.$$

Now using Lemma 4, with α and ϵ as γ and c, respectively, it follows that

$$|z(t) - z(0)| \leq (e^{\alpha t} - 1)|z(0)| + \epsilon \int_0^t e^{\alpha(t-s)} ds.$$

Making the change of variable $u := t - s$, we observe that

$$\int_0^t e^{\alpha(t-s)} ds = -\int_t^0 e^{\alpha u} du = \int_0^t e^{\alpha s} ds,$$

and therefore

$$|z(t)| \leq |z(0)| + |z(t) - z(0)| \leq K(t)|z(0)| + \epsilon \int_0^t K(s) ds,$$

or, equivalently,

$$|\phi(t,x_0,m)-\phi(t,y_0,m')| \leq K(t)|x_0-y_0|+\epsilon\int_0^t K(s)ds.$$

□

With ϕ now satisfying hypotheses 1, let Ψ be the function arising from ϕ using the procedure described in sections 3 and 4.

Theorem 2. *Consider the control system with regulated controls given by a rhs f that satisfies the lipschitz condition (12). Then its transition function Γ and Ψ are one and the same.*

Proof. We first prove that Γ and Ψ agree on piecewise constant controls.

Let $T>0$, $x\in X$, and $u:[0,T]\to M$, a piecewise constant function defined by a finite partition $0=t_0<t_1<t_2<\ldots<t_k=T$ of the interval $[0,T]$, and elements $m_1,m_2,\ldots,m_k$ of M with $u(t)=m_i$ whenever $t\in(t_{i-1},t_i)$, for $i\in\{1,2,\ldots,k\}$.

By Definition 4, we may write:

$$\Psi(T,x,u)=\phi(t_k-t_{k-1},\phi(t_{k-1}-t_{k-2},\ldots,\phi(t_1,x,m_1),m_2),\ldots,m_k).$$

Using equation (10) repeatedly and the semigroup property of Γ, we obtain:

$$\begin{aligned}\Psi(T,x,u)=\ &\phi(t_k-t_{k-1},\phi(t_{k-1}-t_{k-2},\ldots,\Gamma(t_1,x,m_1),m_2),\ldots,m_k)=\\ &\ldots\\ &=\Gamma(t_k-t_{k-1},\Gamma(t_{k-1}-t_{k-2},\ldots,\Gamma(t_1,x,m_1),m_2),\ldots,m_k)=\\ &=\Gamma(T,x,u).\end{aligned}$$

Now let t and x be as before and let u be a regulated control. Lema 2.8.2 in [1] implies that if M is separable and $\Gamma(t,x,u)=z$, then there exists a sequence of piecewise constant controls $\{u_n\}_n$ converging uniformly to u so that, if $z_n:=\Gamma(t,x,u_n)$, then z_n converges to z. This means that

$$\Gamma(t,x,u)=\lim_n \Gamma(t,x,u_n).$$

By the first part of the proof we have

$$\Gamma(t,x,u_n)=\Psi(t,x,u_n).$$

So

$$\Gamma(t,x,u)=\lim_n \Psi(t,x,u_n)=\Psi(t,x,u),$$

by Definition 5. □

This theorem shows how the control system Γ and the dynamical polysystem ϕ are essentially the same thing, as Γ gives rise to ϕ and is then recovered from ϕ under the form of Ψ.

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