A Priori Estimate of the Solution of Quasi-Inverse Problem

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Abstract
In (Raid Almomani and Hasan Almefleh, 2012), we formulated the control problem of heat conduction problem with inverse direction of time and integral boundary conditions and we show the non-well-posedness of this problem. In (Hasan Almefleh, 2013), we reduce the solution of the control problem of the inhomogeneous heat equation to the homogeneous case. In this paper we establish a priori estimate for the solution of quasi-inverse problem. The solution of our problem plays an important role in optimal control in heat conduction theory and in plasma physics, that is, in those problems where we have an integral restriction on a function.

Keywords: Integral boundary conditions, Control problem of heat conduction, Quasi-inverse problem, A priori estimate.

We consider in the domain $Q=\{(x,t):\ 0<x<1,\ 0<t<T\}$ the quasi-inverse problem

$$\frac{\partial U_\epsilon}{\partial t}-\frac{\partial^2 U_\epsilon}{\partial x^2}-\epsilon\frac{\partial^3 U_\epsilon}{\partial x^2 dt}=0,\quad (x,t)\in Q,\qquad t<T$$

$$U_\epsilon(x,T)=X(x),\quad 0\le x\le 1$$

$$U_\epsilon(0,t)=0,\quad \int_0^1 U_\epsilon(x,t)dx=0, 0\le t\le T \qquad (1)$$

It is supposed that the function $X(x)\in W_2^1(0,1)$ and satisfies the consistency conditions

$$X(0)=0,\ \int_0^1 X(x)dx=0, \qquad (2)$$

where $W_2^1(0,1)$ is first order Sobolev space.
To deal with the integration by the increasing variable t, we make the change of t to $T-t$ in (1) and instead of problem (1) we will consider the following equivalent problem

$$\frac{\partial U_\epsilon}{\partial t}+\frac{\partial^2 U_\epsilon}{\partial x^2}-\epsilon\frac{\partial^3 U_\epsilon}{\partial x^2 dt}=0,\quad (x,t)\in Q,\qquad t<0$$

$$U_\epsilon(x,0)=X(x),$$

$$U_\epsilon(0,t)=0,\quad \int_0^1 U_\epsilon(x,t)dx=0, \qquad (3)$$

where $U_\epsilon(x,t)=\ U_\epsilon(x,T-t)$.
We suppose that the solution of (3) exist and we establish a priori estimate for it.
Consider the identity

$$\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial t}+\frac{\partial^2 U_\epsilon}{\partial x^2}-\epsilon\frac{\partial^3 U_\epsilon}{\partial x^2 dt}\right)\left[(1-x)\frac{\partial U_\epsilon}{\partial t}+J\frac{\partial U_\epsilon}{\partial t}\right]dxdt=0 \tag{4}$$

which is derived by scalar multiplication of (3) in $L_2(Q_\tau)$ by the function

$$(1-x)\frac{\partial U_\epsilon}{\partial t}+J\frac{\partial U_\epsilon}{\partial t} \tag{5}$$

where

$$J_g=\int_0^x g(\xi,t)d\xi$$

$$Q_\tau=\{(x,t),\ \ 0<x<1,\ \ 0<t<\tau<T\}$$

Integrating by parts in (4) the terms that contain the second partial derivatives with respect to x and taking into account the boundary conditions for the function $U_\epsilon(x,t)$, we get

$$\int_0^\tau\int_0^1\frac{\partial^2 U_\epsilon}{\partial x^2}\left[(1-x)\frac{\partial U_\epsilon}{\partial t}+J\frac{\partial U_\epsilon}{\partial t}\right]dxdt=-\int_0^\tau\int_0^1(1-x)\frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x\partial t}dxdt,$$

$$-\epsilon\int_0^\tau\int_0^1\frac{\partial^3 U_\epsilon}{\partial x^2 dt}\left[(1-x)\frac{\partial U_\epsilon}{\partial t}+J\frac{\partial U_\epsilon}{\partial t}\right]dxdt=\epsilon\int_0^\tau\int_0^1(1-x)\left(\frac{\partial^2 U_\epsilon}{\partial xdt}\right)^2 dxdt.$$

Thus from (4) we will have the following identity

$$\int_0^\tau\int_0^1(1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt+\epsilon\int_0^\tau\int_0^1(1-x)\left(\frac{\partial^2 U_\epsilon}{\partial xdt}\right)^2 dxdt =\int_0^\tau\int_0^1(1-x)\ \frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x^2}dxdt \tag{6}$$

We estimate the integral in the right hand side through the values in the left hand side with the help of the inequality $\left(ab\le\frac{1}{2}\left(\epsilon a^2+\frac{1}{\epsilon}b^2\right)\right)$, we get

$$\int_0^\tau\int_0^1(1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt+\frac{\epsilon}{2}\int_0^\tau\int_0^1(1-x)\left(\frac{\partial^2 U_\epsilon}{\partial xdt}\right)^2 dxdt \le\frac{1}{2\epsilon}\int_0^\tau\int_0^1(1-x)\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt. \tag{7}$$

Using the equality

$$\int_0^\tau\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dt=\int_0^\tau dt\int_0^t\frac{\partial}{\partial y}\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dy+\int_0^\tau\left(\frac{\partial U_\epsilon}{\partial x}\right)^2\Bigg|_{t=0} dt \tag{8}$$

And taking into account the initial condition (3) we have

$$\int_0^\tau\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dt=\int_0^\tau dt\int_0^t 2\frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x\partial y}dy+\tau(X'(x))^2$$

$$\le\delta\tau\int_0^\tau\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dy+\frac{1}{\delta}\int_0^\tau dt\int_0^t\left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dy+\tau(X'(x))^2.$$

Assuming that $T=\frac{1}{2}$, we estimate

$$\frac{1}{2}\int_0^\tau \left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dt \le 2T\int_0^\tau dt\int_0^t \left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dy + T(X'(x))^2 \qquad (9)$$

Consequently the right hand side of (7) can be estimated above by

$$\frac{1}{2\epsilon}\int_0^\tau\int_0^1 (1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$\le \frac{2T}{\epsilon}\int_0^\tau dt\int_0^\tau\int_0^1 (1-x)\left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dxdy + \frac{T}{\epsilon}\int_0^1 (1-x)\left(X'(x)\right)^2 dx. \qquad (10)$$

And from (7) we get the inequality

$$\int_0^\tau\int_0^1 (1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \frac{\epsilon}{2}\int_0^\tau\int_0^1 (1-x)\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt$$
$$\le \frac{2T}{\epsilon}\int_0^\tau dt\int_0^t\int_0^1 (1-x)\left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dxdy + \frac{T}{\epsilon}\int_0^1 (1-x)\left(X'(x)\right)^2 dx. \qquad (11)$$

From which using Gronwall's lemma and passing on to the limit at $\tau \to T$, we get the following a priori estimation for the solution of (3)

$$\int_Q (1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \frac{\epsilon}{2}\int_Q (1-x)\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt$$
$$\le \frac{T}{\epsilon}e^{\frac{4T^2}{\epsilon^2}}\int_0^1 (1-x)\left(X'(x)\right)^2 dx.$$

References

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