Differential Equations with Small Perturbations Operator Coefficients and Deviating Argument

Ababneh Mousa Salty
Finance and Administrative Department,
BAU, Jordan.
ababneh_65@yahoo.com

Raid Almomani
Department of Mathematics,
Yarmouk University,
21163 Irbid - Jordan. raid@yu.edu.jo

Abstract
In this paper, we prove a theorem about the continuous invertability of the operator in the coefficients of a differential equation with small perturbations and deviating argument. In addition of its independent interest, the study of the considered problem is motivated by the fact that it is a particular case of a more general and interesting problem about the perturbations of linear operator acting on infinite dimensional space. Our argument show clearly the general characteristics of the theory of differential equations with deviating argument.

Keywords: operator coefficients, perturbations, continuous invertability, deviating argument.

1 Introduction

In applied mathematics and mathematical physics, second order differential equations or the equivalent system of two first order equations have a great importance [2]. In recent years many results were achieved in the mathematical theory of differential equations with periodic coefficients and deviating argument. For background information, we refer readers to [1], and [3]-[11], among others.

Consider the following equation

$$L_{po}^2 u(t) \equiv D_t^2 u(t) - \sum_{k=0}^{1} \sum_{j=0}^{m} \left[A_{kj} + A_{kj}(t)\right] D_t^k S_{h_{kj}+h_{kj}(t)} u(t) = f(t), \tag{1}$$

$t \in \mathrm{R}$, with small in some sense variable perturbations $A_{kj}(t)$ and $h_{kj}(t)$, $k = 0,1;\quad j = 0,1,...,m$, where $D_t^k = \frac{1}{i^k}\frac{d^k}{dt^k}$.

Our interest in this equation is supported by the fact that it is a particular case of a more general and more interesting problem about the perturbations of linear operator acting in an infinite dimensional space. We shall show some general characteristics of the theory of differential equations with deviating argument. It is well known from functional analysis that if the operator A is invertible and the norm of the operator B does not exceed the number $\left\|A^{-1}\right\|^{-1}$, then the operator $A + B$ is also invertible. This fact can not be obviously transferred to the case of the operator L_p^2. Really that if we have the equation

$$lu(t) \equiv \left(D_t^2 - A\right)u(t) = f(t), t > t_0 \geq -\infty,$$

with continuous invertible operator $l : X_{(0,\omega)}^{2,0} \to Y_{(0,\omega)}^{0,0}$ and $\left\|B(t)\right\| < \left\|l^{-1}\right\|^{-1}$, then the operator $\left(l + B(t) : X_{(0,\omega)}^{2,0} \to Y_{(0,\omega)}^{0,0}\right)$ is continuously invertible. But, if we have the equation

$$l_p u(t) \equiv \left(D_t^2 - AS_h\right)u(t) = f(t), t > t_0 \geq -\infty,$$

with continuous invertible operator $l_p : X_{(0,\omega)}^{2,0} \to Y_{(0,\omega)}^{0,0}$, then from the analogy of the requirement $\left\|B(t)\right\| < \left\|l^{-1}\right\|^{-1}$ it does not imply the continuous invariability of the operator $l_{po} \equiv D_t^2 - (A + B(t))S_{h+h(t)}$ This fact is regardless the size of the variable component $h(t)$ of deviation argument. To show this, we rewrite (1) in the following form

$$\left(L_p^2 + L_1\right)u(t) = f(t), \qquad (2)$$

$$D_t^2 u(t) - {}_{k=0}^{1}{}_{j=0}^{m}\left[A_{kj} + A_{kj}(t)\right]D_t^k S_{h_{kj}+h_{kj}(t)}u(t)$$
$$= D_t^2 u(t) - {}_{k=0}^{1}{}_{j=0}^{m}\left[A_{kj} + A_{kj}(t)\right]D_t^k S_{h_{kj}+h_{kj}(t)}u(t)$$
$$- {}_{k=0}^{1}{}_{j=0}^{m} A_{kj} D_t^k S_{h_{kj}} u(t)$$
$$= D_t^2 u(t) - {}_{k=0}^{1}{}_{j=0}^{m} A_{kj} D_t^k S_{h_{kj}} u(t) + L_1^2 u(t) = f(t)$$

where

$$L_1 = {}_{k=0}^{1}{}_{j=0}^{m}\left\lfloor A_{kj}\left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right) - A_{kj}(t) S_{h_{kj}+h_{kj}(t)}\right\rfloor D_t^k. \qquad (3)$$

It should be noticed here that the second term in brackets is small enough as a consequence of the fact that the term $\left\|A_{kj}(t)\right\|_Y$ is small. Concerning the first term, we can use the size of $h_{kj}(t)$. But

$$\left\|A_{kj} D_t^k + \left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right)u(t)\right\|_Y$$
$$\leq c\left\|A_{kj}\right\|\left\|D_t\left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right)u(t)\right\|_Y$$
$$= c\left\|A_{kj}\right\|\left\|{}_{t-h_{kj}-h_{kj}(t)}^{t-h_{kj}} u''(s)ds\right\|_X.$$

Since $u''(t) \notin X$, then the norm under the integral sign is impossible and the presence of the deviating argument could be talk about it self. The arising difficulty impose on the operator A_{kj} an extra conditions.

2 The Main Result

Theorem. *Assume that the following conditions hold:*

1. $A_{kj} \in L_0(Y,Y), \quad j = 0,1,...,m, \quad A_{kj} \in L_\infty(X,Y), \quad j = 1,2,...,m, \quad k = 0,1.$

2. $\forall_n \exists R_n, \ \|nR_n\|_X = O(1), \ \|n^2 R_n\|_Y = O(1), \quad n \to \infty.$

3. $f(t) \in Y^{0,0}_{(0,\omega)}$.

Then there exist $\varepsilon > 0$ *such that if conditions* $\|A_{kj}(t)\|_Y \le \varepsilon, \ |h_{kj}(t)| < \varepsilon, \ t \in (0,\omega), \ h_{kj}(t) \in H(0,\omega),$ $j = 0,1,\ldots,m, \ k = 0,1,$ *hold, then the operator* $L^2_{p0} : X^{2,0}_{(0,\omega)} \to Y^{0,0}_{(0,\omega)}$ *is continuously invertible. Here,* X, Y *are Hilbert's spaces,* $X \subset Y$, $\|\cdot\|_X \ge \|\cdot\|_Y, \|\cdot\|_X$ *is a norm in the space* X; $L_\infty(X,Y)$ *is the set of totally continuous operators from* X *to* Y; $L(X,Y)$ *is the set of linear bounded operators from* X *to* Y, *and* $L_0(Y,Y)$ *is the set of closed operators from* Y *to* Y.

Proof. We rewrite (1) in the following form

$$(L^2_p + L_1)u(t) = f(t),$$

where

$$L_1 = \sum_{k=0}^{1}\sum_{j=0}^{m} \left[A_{kj}\left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right) - A_{kj}(t) S_{h_{kj}+h_{kj}(t)} \right] D_t^k .$$

By the imposed conditions, the operator L^2_p is continuously invertible. We show that the operator L^2_p is bounded. Moreover $\|L_1\| \le \left\| \left(L^2_p\right)^{-1} \right\|^{-1}$.

We estimate the following norm

$$\left(\|L_1 u(t)\|^{0,0}_{(0,\omega)}\right)^2 = \left(\left\| \sum_{k=0}^{1}\sum_{j=0}^{m} \left[A_{kj}\left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right) \right.\right.\right.$$

$$\left.\left.\left. - A_{kj}(t) S_{h_{kj}+h_{kj}(t)} \right] D_t^k u(t) \right\|^{0,0}_{0,\omega} \right)$$

$$\le 2\left(\left\| \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj}\left(S_{h_{kj}} - S_{h_{kj}+h_{kj}(t)}\right) D_t^k u(t) \right\|^{0,0}_{(0,\omega)} \right)^2$$

$$+ 2\left(\left\| \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} S_{h_{kj}+h_{kj}(t)} D_t^k u(t) \right\|^{0,0}_{(0,\omega)} \right)^2$$

$$= \Phi_1 + \Phi_2$$

$$\Phi_2 = \frac{2}{\omega} \int_0^\omega \left\| \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} S_{h_{kj}+h_{kj}(t)} D_t^k u(t) \right\|^2_Y dt$$

$$\le \frac{4}{\omega}(m+1) \int_0^\omega \sum_{k=0}^{1}\sum_{j=0}^{m} \|A_{kj}\|^2_Y \left\| S_{h_{kj}+h_{kj}(t)} D_t^k u(t) \right\|^2_X dt$$

$$\le c\varepsilon^2 \sum_{k=0}^{1}\sum_{j=0}^{m} \int_0^\omega \left\| D_t^k u(t - h_{kj} - h_{kj}(t)) \right\|^2_X dt$$

$$= c\varepsilon^2 \sum_{k=0}^{1}\sum_{j=0}^{m} \int_0^\omega \left\| D_t^k u(t - h_{kj} - h_{kj}(t)) \right\|^2_X dt \, \frac{d(t - h_{kj - h_{kj}(t)})}{1 - h_{kj}(t)}$$

$$\le \frac{\varepsilon^2}{1-r}\sum_{k=0}^{1}\sum_{j=0}^{m}\int_{-h_{kj}-h_{kj}(0)}^{\omega-h_{kj}-h_{kj}(\omega)}\left\|D_s^k u(s)\right\|_X^2 ds$$

$$\le c\varepsilon^2\sum_{k=0}^{1}\int_0^{\omega}\left\|D_t^k u(t)\right\|_X^2 dt$$

$$\le c\varepsilon^2\left(\left\|u(t)\right\|_{(0,\omega)}^{2,0}\right)^2.$$

Now, we turn to the first term in the expression of L_1, we get

$$\left\|\sum_{k=0}^{1}\sum_{j=0}^{m}A_{kj}\left(S_{h_{kj}}-S_{h_{kj}+h_{kj}(t)}\right)D_t^k u(t)\right\|_{(0,\omega)}^{0,0}$$

$$\le 2\left(\left\|\sum_{j=0}^{m}A_{0j}\left(S_{h_{0j}}-S_{h_{0j}+h_{0j}(t)}\right)u(t)\right\|_{(0,\omega)}^{0,0}\right)^2$$

$$+2\left(\left\|\sum_{j=0}^{m}A_{1j}\left(S_{h_{1j}}-S_{h_{1j}+h_{1j}(t)}\right)u'(t)\right\|_{(0,\omega)}^{0,0}\right)^2$$

$$=\frac{2}{\omega}\left[\int_0^{\omega}\left\|\sum_{j=0}^{m}A_{0j}\left(S_{h_{0j}}-S_{h_{0j}+h_{0j}(t)}\right)u(t)\right\|_Y^2 dt\right.$$

$$\left.+\int_0^{\omega}\left\|\sum_{j=0}^{m}A_{1j}\left(S_{h_{1j}}-S_{h_{1j}+h_{1j}(t)}\right)u'(t)\right\|_Y^2 dt\right]$$

$$=I_1+I_2$$

Now, we estimate the first integral,

$$I_1\le\frac{2}{\omega}\left\{\varepsilon^2\int_0^{\omega}\left\|\left(S_{h_{0j}}-S_{h_{0j}+h_{0j}(t)}\right)u(t)\right\|_X^2 dt\right.$$

$$\left.+X_{A_{0j}}^2(\varepsilon)\int_0^{\omega}\left\|\left(S_{h_{0j}}-S_{h_{0j}+h_{0j}(t)}\right)u(t)\right\|_Y^2 dt\right\}$$

$$\le c\left\{\varepsilon^2\int_0^{\omega}\left\|u(t-h_{0j})\right\|_X^2+\left\|u(t-h_{0j}-h_{0j}(t)\right\|_X^2 dt\right.$$

$$\left.+X_{A_{0j}}^2(\varepsilon)\int_0^{\omega}\left\|\int_{t-h_{0j}-h_{0j}(t)}^{t-h_{0j}}u'(s)ds\right\|_Y^2 dt\right\}$$

$$\le c\left\{\varepsilon^2\left[\int_{-h_{0j}}^{\omega-h_{0j}}\left\|u(s)\right\|_X^2 ds+\frac{1}{1-r}\int_{-h_{0j}-h_{0j}(0)}^{\omega-h_{0j}-h_{0j}(\omega)}\left\|u(s)\right\|_X^2 ds\right]\right.$$

$$\left.+\int_0^{\omega}\left|\int_{t-h_{0j}-h_{0j}(t)}^{t-h_{0j}}ds\right|\int_{t-h_{0j}-h_{0j}(t)}^{t-h_{0j}}\left\|u'(s)\right\|_Y^2 ds\,dt\right\}$$

$$\le c\left\{\frac{2-r}{1-r}\varepsilon^2\int_0^{\omega}\left\|u(t)\right\|_X^2 dt+t^2\int_0^{\omega}\left\|u'(t)\right\|_Y^2 dt\right\}$$

$$\le c\varepsilon^2\left(\left\|u(t)\right\|_{(0,\omega)}^{2,0}\right)^2.$$

By the same way, we estimate the second integral in the expression for I_2, we get

$$I_2 = c\left\{\frac{2-r}{1-r}\varepsilon^2 \int_0^{\omega} \|u(t)\|_X^2\, dt + c\varepsilon^2 \int_0^{\omega} \|u'(t)\|_Y^2\, dt\right\}$$

$$\leq c\varepsilon^2\left(\|u(t)\|_{(0,\omega)}^{0,2}\right)$$

Thus the following inequality holds

$$\left(\|L_1 u(t)\|_{(0,\omega)}^{0,0}\right)^2 \leq c\varepsilon^2\left(\|u(t)\|_{(0,\omega)}^{2,0}\right)^2.$$

The last inequality implies that

$$\|L_1 u(t)\|_{X_{(0,\omega)}^{2,0}} \to Y_{(0,\omega)}^{0,0} < c\varepsilon^2.$$

Hence, the conclusion of our theorem for certain $\varepsilon > 0$ is a consequence of the Theorem about the operator invariability.+

References

1. Benkhalti, R. and Ezzinbi, K., Periodic solutions for some partial functional differential equations, J. Appl. Math and Stochastic Analysis, 1, (2004), 9-18.

2. Boyce W.E., DiPrima, R.C., Elementary Differential Equations and Boundary Value Problems, Wiley, New York, 1986.

3. Cabada, A., Cid, J.À., On the sign of the Green's function associated to Hill's equation with an indefinite potential, Applied Mathematics and Computation, 205,(1),(2008), 303-308.

4. Huseynov,A., Positive solutions of a nonlinear impulsive equation with periodic boundary conditions, Applied Mathematics and Computation, 217 (1), (2010), 247-259

5. Kiguradze I. and Mukhigulashvili S., On periodic solutions of the system of two linear differential equations Mem. Differential Equations Math. Phys. 48 (2009), 175–182

6. Li, X. Zhang, Z., Periodic solutions for damped differential equations with a weak repulsive singularity, Nonlinear Analysis: Theory, Methods & Applications, 70 (6), (2009), 2395-2399.

7. Nieto, J.J. and Rodrguez-López, R., Green's function for second-order periodic boundary value problems with piecewise constant arguments, Journal of Mathematical Analysis and Applications, 304 (1), (2005), 33-57.

8. Piao, D., Periodic and almost periodic solutions for some differential equations with reflection of the argument, Nonlinear Analysis, 57, (2004), 633-637.

9. Seidov Z B, "A boundary value problem for differential equations with deviating argument", Differential Equations, 12 (1976), 397–400.

10. Terent'ev A G and T Ya Shishova, "On some methods of investigation of equations with deviating argument", Differential Equations, 9 (1973), 978–982.

11. Zhang, Z. and Junyu Wang, J., On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations, Journal of Mathematical Analysis and Applications, 281 (1), (2003), 99-107.