Oscillation properties of third order nonlinear delay difference Equations

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ABSTRACT
In this paper we shall investigate the oscillatory properties of the third order nonlinear difference equations with delay of the form

$$\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)+\sum_{i=1}^{n}q_i(n)\,f\left(x(g_i(n))\right)=0.$$

Whereα_1,α_2 are quotient of odd positive integers. Appling suitable comparison theorems and by a riccati transformation technique we establish some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results improve and extend some known results in the literature. Examples are given to illustrate the importance of the results.

Keywords:-Difference equation, Oscillatory solutions, Delay.

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1. INTRDUCTION

By comparison with some first difference equations whose oscillatory characters are known and by means of aRiccati transformation technique,we obtain several new sufficient conditions for the oscillation of all solutions of the nonlinear difference equation with Delayof the form

$$\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)+\sum_{i=1}^{n}q_i(n)\,f\left(x(g_i(n))\right)=0,\qquad n\geq n_0\,. \tag{1.1}$$

Where$n_0\in N$is a fixed integer, Δdenotes the forward difference operator defined by$\Delta x(n)=x(n+1)-x(n)$ and $\Delta^i x(n)=\Delta\left(\Delta^{i-1}x(n)\right)$. TheReal sequences $\{a(n)\}_{n=n_0}^{\infty},\{b(n)\}_{n=n_0}^{\infty},\{q_i(n)\}_{n=n_0}^{\infty}$, $\{g_i(n)\}_{n=n_0}^{\infty}$and the function f satisfy the following conditions:

$(A_1) a(n), b(n) > 0$ and $q_i(n) > 0\ for\ n \in N(n_0), i = 1,2,\dots,n.$
$(A_2) \alpha_1, \alpha_2$ are quotient of odd positive integers.
$(A_3) f \in C(R,R)$ such that $xf(x) > 0, f'(x) > 0\ for\ all\ x \neq 0.$
$(A_4) g_i: N(n_0) \to Z$ Satisfy $g_i(n) < n, \Delta g_i(n) \geq 0 for\ n \in N(n_0)$ and $\lim_{n\to\infty} g_i(n) = \infty, i = 1,2,\dots,n.$

In addition, we will make use of the following conditions:

$$(S_1) - f(-xy) \geq f(xy) \geq f(x)f(y)\ for\ xy > 0.$$

$(S_2) f(u)/u^{\alpha} \geq K > 0, K$is a real constant, $u > 0.$
$(S_3)\ f(u) - f(v) = B(u,v)(u - v)\ \ for\ u, v \neq 0\ .$
Where B is a nonnegative real valued function, and
$f^{\frac{1}{\alpha}-1}(u)B(u,v) \geq \mu > 0\ \ for\ u, v \neq 0\ and\ \mu\ is\ a\ constant\ .$
By a solution of equation (1.1) we mean a nontrivial sequence $\{x(n)\}$ satisfying equation (1.1)for all $n \geq n_0$. A solution $\{x(n)\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation(1.1) is called oscillatory if all its solutions are oscillatory.
In recent years, there has been an increasing interest in the study of the problem ofdeterminingtheoscillationandnonoscillationofsolutionsofdifferenceequations(see, e.g., [l-20] and the references cited therein).Our aim in this paperis to present some new oscillation criteria for equation (1.1),Our results improve and expanded some known results, for example, the results obtained by [Selvaraj et al. (2012), (2010), Saker (2010), Graceet al. (2009), Schmeidal(2006), Graef et al. (1999)] and the references cited therein.In this paper, the details of the proofs of results for nonoscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions. The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems and in Section 3, we provide some examples to illustrate the main results.

2. MAIN RESULTS

In this section, we establish some new oscillation criteria for the equation (1.1) under the following conditions

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) = \infty\ ,\ \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) = \infty\ \ .(2.1)$$

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) < \infty\ ,\ \ \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) = \infty\ \ .(2.2)$$

$$\sum_{n=n_0}^{\infty} a^{-\frac{1}{\alpha_2}}(n) < \infty\ ,\ \ \sum_{n=n_0}^{\infty} b^{-\frac{1}{\alpha_1}}(n) < \infty\ \ .(2.3)$$

We begin with some useful lemmas, which will be used in obtaining our main results. We Let

$$g(n) = min\{g_1(n), g_2(n), \dots, g_n(n)\},\ \ Q(n) = \sum_{i=1}^{n} q_i(n),$$

$$\delta_1(g(n), n_2) = \sum_{s=n_2}^{g(n)-1} a^{-\frac{1}{\alpha_2}}(s)\ , \delta(n) = \sum_{u=n}^{\infty} a^{-\frac{1}{\alpha_2}}(u)\ , \Psi(n) = KQ(n)\left(\sum_{s=n_2}^{g(n)-1} \frac{1}{\left(b(s)\right)^{\frac{1}{\alpha_1}}}\right)^{\boldsymbol{\alpha}}.$$

Lemma 2.1. Let $\{x(n)\}$ be an eventually positive solution of the equation (1.1) and suppose that $\{x(n)\}$ satisfies

$\Delta x(n) > 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0, \Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) \leq 0 \; for \; all \; n \geq N.$

Then there exists $n \geq n_1$such that

$$\Delta x(n) \geq b^{-\frac{1}{\alpha_1}}(n)\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha}}\left(\sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}, \tag{2.4}$$

where

$\alpha := \alpha_1\alpha_2$.

Proof. Since $\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) \leq 0$, we have $a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}$ is non-increasing. Then, we obtain,

$$b(n)\left(\Delta x(n)\right)^{\alpha_1} = b(n_2)\left(\Delta x(n_2)\right)^{\alpha_1} + \sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\left(a(s)\left(\Delta\left(b(s)\left(\Delta x(s)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}$$

$$\geq \left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}} \sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s).$$

It follows that

$$\Delta x(n) \geq b^{-\frac{1}{\alpha_1}}(n)\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha_1\alpha_2}}\left(\sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}.$$

The proof is complete. ■

Lemma 2.2. Assume that (2.1) holds.Let $\{x(n)\}$ be an eventually positive solution of equation (1.1).Then,for sufficiently large n, there are only two possible cases:

(I):$\Delta x(n) > 0$,$\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0$,

(II)**:**$\Delta x(n) < 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0$.

Proof. Pick $n_1 \geq n_0$such that$x\left(g(n)\right) > 0$, for $n \geq n_1$. Since $\{x(n)\}$ is an eventually positive solution of (1.1). From equation (1.1), (A_1) *and* (A_3) we see that $\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) \leq 0, for \; all \; n \geq n_1$. Then, $a(n)\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)$ is non-increasing sequence and thus $\Delta x(n)$ and $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)$ are eventually of one sign. There are the following four possibilities to consider

(I): $\Delta x(n) > 0$,$\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0 \; for \; all \; large \; n,$

(II): $\Delta x(n) < 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0 \; for \; all \; large \; n,$

(III): $\Delta x(n) > 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) < 0 \; for \; all \; large \; n,$ and

(IV):$\Delta x(n) < 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) < 0 \; for \; all \; large \; n.$

We claim that $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) > 0$. If not, then, (III)or (IV)hold**.**If (III) holds, then,$b(n)\left(\Delta x(n)\right)^{\alpha_1}$ is strictly decreasing and there exists a negative constant M such that

$a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2} < M \; for \; all \; n \geq n_2.$

Dividing by $a(n)$ and Summing the above inequality from $n_2 \ ton - 1$, we obtain

$$b(n)\big(\Delta x(n)\big)^{\alpha_1} \le b(n_2)\big(\Delta x(n_2)\big)^{\alpha_1} + M^{\frac{1}{\alpha_2}} \sum_{s=n_2}^{n-1} \big(a(s)\big)^{-\frac{1}{\alpha_2}}.$$

Letting $n \to \infty$, and using (2.1) then $b(n)\big(\Delta x(n)\big)^{\alpha_1} \to -\infty$, which contradicts that $\Delta x(n) > 0$. If (IV) holds, then,

$$b(n)\big(\Delta x(n)\big)^{\alpha_1} \le b(n_2)\big(\Delta x(n_2)\big)^{\alpha_1} = K < 0.$$

Dividing by $b(n)$ and Summing the above inequality from $n_2 \ ton - 1$, we obtain

$$x(n) \le x(n_2) + K^{\frac{1}{\alpha_1}} \sum_{s=n_2}^{n-1} \big(b(s)\big)^{-\frac{1}{\alpha_1}}.$$

Letting $n \to \infty$, and using (2.1) then$x(n) \to -\infty$, which contradicts the fact that $x(n) > 0$. Then, we have $\Delta\big(b(n)\big(\Delta x(n)\big)^{\alpha_1}\big) > 0 \ for \ n \ge n_1$. And thus either$\Delta x(n) > 0 \ or \ \Delta x(n) < 0$.The proof is complete.■

Lemma 2.3. Assume that(2.1)holds.Let $\{x(n)\}$ be an eventually positive solution of theequation (1.1) and suppose that (II) of Lemma 2.2 holds. If

$$\sum_{v=n_0}^{\infty} \left(b^{-\frac{1}{\alpha_1}}(v) \left(\sum_{u=n}^{\infty} a^{-\frac{1}{\alpha_2}}(u) \left(\sum_{s=n}^{\infty} Q(s) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \right) = \infty. \tag{2.5}$$

Then $x(n) \to 0 \ as \ n \to \infty$.

Proof. Pick $n_1 \ge n_0$such that$x\big(g(n)\big) > 0$, for $n \ge n_1$. Since $\{x(n)\}$ is a positive decreasing solution of equation (1.1). Then $\lim_{n\to\infty} x(n) = l_1 \ge 0$.Assume that$l_1 > 0$ then $x\big(g_i(n)\big) \ge l_1$ for $n \ge n_2 \ge n_1$.From equation (1.1) , we have

$$0 \ge \Delta\left(a(n)\left(\Delta\big(b(n)\big(\Delta x(n)\big)^{\alpha_1}\big)\right)^{\alpha_2}\right) + f(l_1)Q(n). \tag{2.6}$$

Summing equation (2.6)from $n \ to\infty$, we obtain

$$a(n)\left(\Delta\big(b(n)\big(\Delta x(n)\big)^{\alpha_1}\big)\right)^{\alpha_2} \ge f(l_1) \sum_{s=n}^{\infty} Q(s).$$

It follows that

$$\Delta\big(b(n)\big(\Delta x(n)\big)^{\alpha_1}\big) \ge \left(\frac{f(l_1)}{a(n)}\right)^{\frac{1}{\alpha_2}} \left(\sum_{s=n}^{\infty} Q(s)\right)^{\frac{1}{\alpha_2}}. \tag{2.7}$$

Summing the above inequality from $n \ to\infty$, we find

$$-\Delta x(n) \ge \frac{f^{\frac{1}{\alpha}}(l_1)}{b^{\frac{1}{\alpha_1}}(n)} \left(\sum_{u=n}^{\infty} a^{-\frac{1}{\alpha_2}}(u) \left(\sum_{s=n}^{\infty} Q(s) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}}. \tag{2.8}$$

Summing the above inequality from $n_2 \ to\infty$, we find

$$x(n_2) \ge f^{\frac{1}{\alpha}}(l_1) \sum_{v=n_2}^{\infty} b^{-\frac{1}{\alpha_1}}(v) \left(\sum_{u=n}^{\infty} a^{-\frac{1}{\alpha_2}}(u) \left(\sum_{s=n}^{\infty} Q(s) \right)^{\frac{1}{\alpha_2}} \right)^{\frac{1}{\alpha_1}}.$$

This contradicts condition (2.5). Then $\lim_{n\to\infty} x(n) = 0$. ■

2.1. Nonexistence of solutions of type (I)

Next, we shall establish some criteria for the nonexistence of solution of type (I) for equation (1.1).

Theorem 2.1. Let $(S_1), (A_1) - (A_4)$ hold. If the first order delay equation

$$\Delta(y_n) + Q(n) f\left(y^{\frac{1}{\alpha}}(g(n))\right) f\left(\sum_{s=n_0}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_0}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right) = 0, \qquad (2.9)$$

is oscillatory, then equation (1.1) has no solution of type (I).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $n_0 \in N$ such that (I) holds for $n \geq n_0$. From Lemma (2.1), we have

$$\Delta x(n) \geq b^{-\frac{1}{\alpha_1}}(n) y^{\frac{1}{\alpha}}(n)\left(\sum_{s=n_2}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}.$$

where $y(n) = a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}$. Summing the above inequality from n_2 to $n-1$, we obtain

$$x(n) \geq \sum_{s=n_2}^{n-1} b^{-\frac{1}{\alpha_1}}(s) y^{\frac{1}{\alpha}}(s)\left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}$$

$$\geq y^{\frac{1}{\alpha}}(n)\sum_{s=n_2}^{n-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}.$$

There exists a $n_3 \geq n_2$ with $(n) \geq n_2$ $for\ all\ n \geq n_3$, such that

$$x(g(n)) \geq y^{\frac{1}{\alpha}}(g(n))\sum_{s=n_2}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}.$$

From equation (1.1), (S_1) and the last inequality, we obtain, $for\ n \geq n_3$,

$$-\Delta y(n) \geq f\left(x(g(n))\right) Q(n)$$

$$\geq Q(n) f\left(y^{\frac{1}{\alpha}}(g(n))\right) f\left(\sum_{s=n_2}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_2}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right).$$

Summing the last inequality from n to ∞, we get

$$y(n) \geq \sum_{s=n}^{\infty} Q(s) f\left(y^{\frac{1}{\alpha}}(g(s))\right) f\left(\sum_{v=n_2}^{g(s)-1} b^{-\frac{1}{\alpha_1}}(v)\left(\sum_{u=n_2}^{v-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right).$$

The sequence $\{y(n)\}$ is obviously strictly decreasing. Hence, by the discrete analog of Theorem1 in [14], we conclude that there exists a positive solution $\{y(n)\}$ of equation (2.9) which tends to zero this contradicts that (2.9) is oscillatory. The proof is complete. ■

Corollary 2.1. Let $(S_2), (A_1) - (A_4)$ hold. If the first order delay equation

$$\Delta(y_n) + KQ(n)y(g(n))\left(\sum_{s=n_0}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_0}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right)^{\alpha} = 0,$$

is oscillatory, then equation (1.1) has no solution of the type (I).

Theorem 2.2. Let $(S_3), (A_1) - (A_4)$ hold. Further, assume that there exists a positive nondecreasing sequence $\{\rho(n)\}$, if

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1}\left(\rho(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(s))^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\mu\rho(s)\delta_1^{\frac{1}{\alpha_1}}(g(s),n_2)\right)^{\alpha}}\right) = \infty. \tag{2.10}$$

Then equation (1.1) has no solution of the type (I).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $n_0 \in N$ such that (I) holds for $n \geq n_0$. Define the sequence $\omega(n)$ by

$$\omega(n) := \rho(n)\frac{a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2}}{f\left(x(g(n))\right)}. \tag{2.11}$$

Then $\omega_n > 0$. From (2.11) and (S_3), we have

$$\Delta\omega(n) = \Delta\rho(n)\frac{a(n+1)\Delta\big(b(n+1)(\Delta x(n+1))^{\alpha_1}\big)^{\alpha_2}}{f\left(x(g(n+1))\right)} + \rho(n)\Delta\left(\frac{a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2}}{f\left(x(g(n))\right)}\right)$$

$$= \Delta\rho(n)\frac{a(n+1)\Delta\big(b(n+1)(\Delta x(n+1))^{\alpha_1}\big)^{\alpha_2}}{f\left(x(g(n+1))\right)} + \rho(n)\frac{\Delta\left(a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2}\right)}{f\left(x(g(n))\right)}$$

$$-\rho(n)\frac{a(n+1)\Delta\big(b(n+1)(\Delta x(n+1))^{\alpha_1}\big)^{\alpha_2}\Delta\left(f\left(x(g(n))\right)\right)}{f\left(x(g(n+1))\right)f\left(x(g(n))\right)}.$$

$$= \Delta\rho(n)\frac{a(n+1)\Delta\big(b(n+1)(\Delta x(n+1))^{\alpha_1}\big)^{\alpha_2}}{f\left(x(g(n+1))\right)} + \rho(n)\frac{\Delta\left(a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2}\right)}{f\left(x(g(n))\right)}$$

$$-\rho(n)\frac{a(n+1)\Delta\big(b(n+1)(\Delta x(n+1))^{\alpha_1}\big)^{\alpha_2}B\left(x(g(n+1)),x(g(n))\right)}{f\left(x(g(n+1))\right)f\left(x(g(n))\right)}\Delta\left(x(g(n))\right). \tag{2.12}$$

From *Lemma* 2.1, there exists $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$ such that

$$\Delta x(g(n)) \geq \left(a(g(n))\left(\Delta\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha}} b^{-\frac{1}{\alpha_1}}(g(n))\left(\sum_{s=n_2}^{g(n)-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}. \tag{2.13}$$

Since $\Delta\left(a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2}\right) \leq 0, g(n) < n$, we get

$$a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2} \leq a(g(n))\left(\Delta\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)\right)^{\alpha_2}. \tag{2.14}$$

Then it follows that

$$a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2} \le a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}. \tag{2.15}$$

It follows from (2.12) that

$$\Delta\omega(n) \le \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) + \rho(n)\frac{\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)}{f\left(x(g(n))\right)}$$

$$-\rho(n)\frac{a(n+1)\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)^{\alpha_2} B\left(x(g(n+1)), x(g(n))\right)}{f\left(x(g(n+1))\right) f\left(x(g(n))\right)}$$

$$\times\left(a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\frac{1}{\alpha}} b^{-\frac{1}{\alpha_1}}(g(n))\delta_1^{\frac{1}{\alpha_1}}(g(n), n_2) \ . \tag{2.16}$$

From (1.1), (2.11), $(S_3)$and (2.16), we have

$$\Delta\omega(n) \le -\rho(n)Q(n) + \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)$$

$$-\mu\rho(n)\frac{\delta_1^{\frac{1}{\alpha_1}}(g(n), n_2) b^{-\frac{1}{\alpha_1}}(g(n))}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)}\omega^{\frac{\alpha+1}{\alpha}}(n+1). \tag{2.17}$$

Using (2.17) and the inequality

$$Bu - Au^{\alpha+1/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, A > 0, \tag{2.18}$$

We have

$$\Delta\omega(n) \le -\rho(n)Q(n) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{\left(\Delta\rho(n)\right)^{\alpha+1} b^{\alpha_2}(g(n))}{\left(\mu\rho(n)\delta_1^{\frac{1}{\alpha_1}}(g(n), n_2)\right)^\alpha}.$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1}\left(\rho(s)Q(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{\left(\Delta\rho(s)\right)^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\mu\rho(s)\delta_1^{\frac{1}{\alpha_1}}(g(s), n_2)\right)^\alpha}\right) \le \omega(n_2).$$

Whichis contrary to(2.10).This completes the proof of Theorem 2.2. ■

Theorem 2.3. Assume that $(S_3), (A_1) - (A_4)$hold. Let$\{\rho(n)\}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H(m,n) | m \ge n \ge 0\}$ and $h(m,n)$ such that

(i)$H(m,m) = 0\ for\ m \ge 0,$

$(ii) H(m,n) > 0\ for\ m > n > 0,$

$(iii)\Delta_2 H(m,n) = H(m,n+1) - H(m,n) \le 0\ for\ m > n \ge 0,$

(iv)$h(m,n) = -\frac{\Delta_2 H(m,n)}{\sqrt{H(m,n)}}$.If

$$\lim_{m\to\infty}\sup\frac{1}{H(m,0)}\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n) - \lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{\left(\varphi(n)\right)^\alpha}\right) = \infty, \tag{2.19}$$

Where

$$\varphi(n)=\frac{\mu\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)b^{\frac{1}{\alpha_1}}\big(g(n)\big)}\delta_1^{\frac{1}{\alpha_1}}(g(n),n_2),\quad \vartheta(m,n)=\left(\frac{\Delta\rho(n)}{\rho(n+1)}-\frac{h(m,n)}{\sqrt{H(m,n)}}\right),\lambda=\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}.$$

Then equation (1.1) has no solution of the type (I).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of the type (I). Then, there is a $n_0\in N$such that(I) holds for$n\geq n_0$. From the proof of Theorem 2.2 we find that (2.17) holds for all $n\geq n_2$. From(2.17), we have

$$\rho(n)Q(n)\leq-\Delta\omega(n)+\frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)-\frac{\mu\rho(n)}{\rho^{\frac{\alpha+1}{\alpha}}(n+1)b^{\frac{1}{\alpha_1}}\big(g(n)\big)}\delta_1^{\frac{1}{\alpha_1}}(g(n),n_2)\omega^{\frac{\alpha+1}{\alpha}}(n+1).\qquad(2.20)$$

Therefore, we have

$$\sum_{n=k}^{m-1}H(m,n)\rho(n)Q(n)\leq-\sum_{n=k}^{m-1}H(m,n)\,\Delta\omega(n)+\sum_{n=k}^{m-1}H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1)-\sum_{n=k}^{m-1}H(m,n)\varphi(n)\omega^{\frac{\alpha+1}{\alpha}}(n+1),$$

which yields after summing by parts

$$\sum_{n=k}^{m-1}H(m,n)\rho(n)Q(n)\leq H(m,k)\omega(k)+\sum_{n=k}^{m-1}\left(\Delta_2H(m,n)+H(m,n)\frac{\Delta\rho(n)}{\rho(n+1)}\right)\omega(n+1)-\sum_{n=k}^{m-1}H(m,n)\varphi(n)\omega^{\frac{\alpha+1}{\alpha}}(n+1).$$

$$\sum_{n=k}^{m-1}H(m,n)\rho(n)Q(n)\leq H(m,k)\omega(k)+\sum_{n=k}^{m-1}\vartheta(m,n)H(m,n)\omega(n+1)-\sum_{n=k}^{m-1}H(m,n)\varphi(n)\omega^{\frac{\alpha+1}{\alpha}}(n+1)$$

From(2.18), we have

$$\sum_{n=k}^{m-1}H(m,n)\rho(n)Q(n)\leq H(m,k)\omega(k)+\sum_{n=k}^{m-1}\lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{\big(\varphi(n)\big)^\alpha}.$$

Then,

$$\sum_{n=k}^{m-1}\left(H(m,n)\rho(n)Q(n)-\lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{\big(\varphi(n)\big)^\alpha}\right)\leq H(m,k)\omega(k)\leq H(m,0)|\omega(k)|.$$

Hence,

$$\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n)-\lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{\big(\varphi(n)\big)^\alpha}\right)\leq H(m,0)\left\{\sum_{n=0}^{k-1}|\rho(n)Q(n)|+|\omega(k)|\right\}.$$

Hence,

$$\lim_{m\to\infty}\sup\frac{1}{H(m,0)}\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n)-\lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{\big(\varphi(n)\big)^\alpha}\right)$$

$$\leq \sum_{n=0}^{k-1} |\rho(n)Q(n)| + |\omega(k)| < \infty,$$

which is contrary to (2.19).This completes the proof of Theorem 2.3. ■

Theorem 2.4. Let $(S_2), (A_1) - (A_4)$ hold. Further, assume that thereexists a positive nondecreasing sequence $\{\rho(n)\}$, if

$$\limsup_{n\to\infty} \frac{1}{n^r} \sum_{s=n_0}^{n-1} (n-s)^r \left(K\rho(s)Q(s) - \frac{1}{2^{3-\alpha}\rho(s)} \frac{\left(\Delta\rho(s)\right)^2 b^{\alpha_2}\left(g(s)\right)}{\delta_1^{\alpha_2}\left(g(s), n_2\right)} \right) = \infty. \quad (2.21)$$

Then equation (1.1) has no solution of type (I).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $n_0 \in N$such that(I) holds for$n \geq n_0$.Define the sequence $\omega(n)$ by

$$\omega(n) := \rho(n) \frac{a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}}{x^{\alpha}\left(g(n)\right)}. \quad (2.22)$$

Then$\omega_n > 0$. From (2.22), we have

$$\Delta\omega(n) = \Delta\rho(n) \frac{a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2}}{x^{\alpha}\left(g(n+1)\right)} + \rho(n)\Delta\left(\frac{a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}}{x^{\alpha}\left(g(n)\right)} \right)$$

$$= \Delta\rho(n) \frac{a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2}}{x^{\alpha}\left(g(n+1)\right)} + \rho(n) \frac{\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)}{x^{\alpha}\left(g(n)\right)}$$

$$-\rho(n) \frac{a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2} \Delta\left(x^{\alpha}\left(g(n)\right)\right)}{x^{\alpha}\left(g(n+1)\right)x^{\alpha}\left(g(n)\right)}.$$

Now, by using the inequality

$x^{\alpha} - y^{\alpha} \geq 2^{1-\alpha}(x-y)^{\alpha}$ *for all* $x \geq y > 0$ *and* $\alpha \geq 1$,

then, we have

$$\Delta\left(x^{\alpha}\left(g(n)\right)\right) = x^{\alpha}\left(g(n+1)\right) - x^{\alpha}\left(g(n)\right) \geq 2^{1-\alpha}\left(x\left(g(n+1)\right) - x\left(g(n)\right)\right)^{\alpha}$$

$$= 2^{1-\alpha}\left(\Delta x\left(g(n)\right)\right)^{\alpha}, \alpha \geq 1.$$

Thus

$$\Delta\omega(n) = \Delta\rho(n) \frac{a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2}}{x^{\alpha}\left(g(n+1)\right)} + \rho(n) \frac{\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)}{x^{\alpha}\left(g(n)\right)}$$

$$-2^{1-\alpha}\rho(n) \frac{a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2}\left(\Delta x\left(g(n)\right)\right)^{\alpha}}{x^{\alpha}\left(g(n+1)\right)x^{\alpha}\left(g(n)\right)}. \quad (2.23)$$

From *Lemma*2.1, there exists $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$ such that

$$\left(\Delta x\left(g(n)\right)\right)^{\alpha} \geq \left(a\left(g(n)\right)\left(\Delta\left(b\left(g(n)\right)\left(\Delta x\left(g(n)\right)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) b^{-\alpha_2}\left(g(n)\right) \left(\sum_{s=n_2}^{g(n)-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\alpha_2}.$$

Since$\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) \leq 0, g(n) < n$, we get

$$a(n+1)\left(\Delta\left(b(n+1)\left(\Delta x(n+1)\right)^{\alpha_1}\right)\right)^{\alpha_2} \leq a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}$$

$$\leq a(g(n))\left(\Delta\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)\right)^{\alpha_2}.$$

From(2.23) and the above inequality, we obtain

$$\Delta\omega(n) \leq \frac{\Delta\rho(n)}{\rho(n+1)}\omega(n+1) + \rho(n)\frac{\Delta\left(a(n)\left(\Delta\left(b(n)(\Delta x(n))^{\alpha_1}\right)\right)^{\alpha_2}\right)}{x^{\alpha}(g(n))}$$

$$-2^{1-\alpha}\frac{\rho(n)}{\rho^2(n+1)b^{\alpha_2}(g(n))}\delta_1^{\alpha_2}(g(s),n_2)\omega^2(n+1). \qquad (2.24)$$

From (1.1), (S_2), (2.24) and the inequality $Bu - Au^2 \leq \frac{B^2}{4A}, A > 0$ in (2.24), we have

$$\Delta\omega(n) \leq -K\rho(n)Q(n) + \frac{1}{2^{3-\alpha}\rho(n)}\frac{(\Delta\rho(n))^2 b^{\alpha_2}(g(n))}{\delta_1^{\alpha_2}(g(n),n_2)}. \qquad (2.25)$$

From (2.25) for $n \geq n_2$, we obtain

$$\sum_{s=n_2}^{n-1}(n-s)^r\left(K\rho(s)Q(s) - \frac{1}{2^{3-\alpha}\rho(s)}\frac{(\Delta\rho(s))^2 b^{\alpha_2}(g(s))}{\delta_1^{\alpha_2}(g(s),n_2)}\right) \leq -\sum_{s=n_2}^{n-1}(n-s)^r\,\Delta\omega(s).$$

Since

$$\sum_{s=n_2}^{n-1}(n-s)^r\,\Delta\omega(s) = r\sum_{s=n_2}^{n-1}(n-s)^{r-1}\,\omega(s) - (n-n_2)^r\omega(n_2)$$

Then, we have

$$\frac{1}{n^r}\sum_{s=n_2}^{n-1}(n-s)^r\left(K\rho(s)Q(s) - \frac{1}{2^{3-\alpha}\rho(s)}\frac{(\Delta\rho(s))^2 b^{\alpha_2}(g(s))}{\delta_1^{\alpha_2}(g(s),n_2)}\right)$$

$$\leq \left(\frac{n-n_2}{n}\right)^r\omega(n_2) - \frac{r}{n^r}\sum_{s=n_2}^{n-1}(n-s)^{r-1}\,\omega(s).$$

Hence,

$$\frac{1}{n^r}\sum_{s=n_2}^{n-1}(n-s)^r\left(K\rho(s)Q(s) - \frac{1}{2^{3-\alpha}\rho(s)}\frac{(\Delta\rho(s))^2 b^{\alpha_2}(g(s))}{\delta_1^{\alpha_2}(g(s),n_2)}\right) \leq \left(\frac{n-n_2}{n}\right)^r\omega(n_2).$$

Hence,

$$\limsup_{n\to\infty}\frac{1}{n^r}\sum_{s=n_2}^{n-1}(n-s)^r\left(K\rho(s)Q(s) - \frac{1}{2^{3-\alpha}\rho(s)}\frac{(\Delta\rho(s))^2 b^{\alpha_2}(g(s))}{\delta_1^{\alpha_2}(g(s),n_2)}\right) \leq \omega(n_2),$$

which is contrary to (2.21).This completes the proof of Theorem 2.4. ■

2.2. Nonexistence of solutions of type (II)

Next, we shall establish some criteria for the nonexistence of solution of type (II) for equation (1.1).

Theorem 2.5. Assume that(S_1), $(A_1)-(A_4)$hold, and there exist two sequences $\xi(n)$ *and* $\eta(n)$ such that $\Delta\xi(n) \geq 0, \xi(n) > n$ *and* $\eta(n) = g\left(\xi(\xi(n))\right) < n$. (2.26)

If the first order delay equation

$$\Delta(x_n) + b^{-\frac{1}{\alpha_1}}(n) f^{\frac{1}{\alpha}}\left(x(\eta(n))\right)\left(\sum_{s=n}^{\xi(n)-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{u=n}^{\xi(s)-1} Q(u)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}} = 0, \qquad (2.27)$$

is oscillatory,then equation (1.1) has no solution of the type (II).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of the type (II). Then, there is a $n_0 \in N$such thattype (II)holds for$n \geq n_0$.Summing equation (1.1) from$n$ to $\xi(n)-1$, we obtain

$$a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2} \geq \sum_{s=n}^{\xi(n)-1} Q(s) f\left(x\left(g(s)\right)\right).$$

Using (2.26) and (S_1), we get

$$\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) \geq a^{-\frac{1}{\alpha_2}}(n) f^{\frac{1}{\alpha_2}}\left(x\left(g\left(\xi(n)\right)\right)\right)\left(\sum_{s=n}^{\xi(n)-1} Q(s)\right)^{\frac{1}{\alpha_2}}.$$

Summing again the above inequality from n to$\xi(n)-1$, we find

$$-b(n)\left(\Delta x(n)\right)^{\alpha_1} \geq \sum_{u=n}^{\xi(n)-1} a^{-\frac{1}{\alpha_2}}(u) f^{\frac{1}{\alpha_2}}\left(x\left(g\left(\xi(u)\right)\right)\right)\left(\sum_{s=n}^{\xi(u)-1} Q(s)\right)^{\frac{1}{\alpha_2}}.$$

It follows that

$$-\Delta x(n) \geq f^{\frac{1}{\alpha}}\left(x(\eta(n))\right) b^{-\frac{1}{\alpha_1}}(n)\left(\sum_{u=n}^{\xi(n)-1} a^{-\frac{1}{\alpha_2}}(u)\left(\sum_{s=n}^{\xi(u)-1} Q(s)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}.$$

Finally, summing the above inequality from$n \ to \ \infty$, we have

$$x(n) \geq f^{\frac{1}{\alpha}}\left(x(\eta(n))\right)\sum_{v=n}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(v)\left(\sum_{u=n}^{\xi(v)-1} a^{-\frac{1}{\alpha_2}}(u)\left(\sum_{s=n}^{\xi(u)-1} Q(s)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right).$$

The sequence$x(n)$is obviously strictly decreasing. Hence, by the discrete analog of Theorem1 in [14],we conclude that there exists a positive solution of equation (2.27) which tends to zero this contradicts that (2.27) is oscillatory.The proof is complete.■

2.3. Oscillation criteriaunder condition (2.1)

Next, we shall establish some oscillation criteria for equation (1.1)under condition (2.1).

Theorem 2.6. Let (2.1), (J_1) and (2.5) hold.Where (J_1): (S_1) and (2.9)hold. Then equation (1.1) is oscillatoryor $\lim_{n\to\infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$such that$x(n) > 0$ and $x(g(n)) > 0$.From the proof of Lemma 2.2$x(n)$ is either of type (I)or (II). From Theorem (2.1), $x(n)$ is not of type (I). From Lemma (2.3), we have $\lim_{n\to\infty} x(n) = 0$.The proof is complete.■

The proof each of the following corollary is similar to that of Theorem 2.6 and hence the details are omitted.

Corollary 2.2. Let $(2.1), (\mathrm{J}_2) and$ (2.5) hold.Where (J_2): (S_3) $and(2.10)$ hold.Then equation (1.1) is oscillatoryor $\lim_{n\to\infty} x(n) = 0$.

Remark2.1. If $\alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1, g(n) \equiv g(n) \equiv n - m + 1$.Then Corollary 2.2 reduced to a special case of Theorem 2 in [7].

Corollary 2.3. Let $(2.1), (\mathrm{J}_3) and$ (2.5) hold.Where (J_3): (S_3) $and(2.19)$ hold.Then equation (1.1) is oscillatoryor $\lim_{n\to\infty} x(n) = 0$.

Remark2.2. If $\alpha_2 \equiv 1, n \equiv 1, g(n) \equiv n - \sigma$. Then Corollary 2.3 extended and improved Theorem 6 in [15].

Remark2.3. If $b(n) \equiv 1$, $\alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1$. Then Corollary 2.3reduced to a special case of Theorem 1 in [18].

Corollary 2.4. Let $(2.1), (\mathrm{J}_4) and$ (2.5) hold.Where (J_4): (S_2) $and(2.21)$ hold.Then equation (1.1) is oscillatoryor $\lim_{n\to\infty} x(n) = 0$.

Remark2.4. If $\alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1, g(n) \equiv n + l$. Then Corollary 2.4extended and improved Theorem 3 in [16].

Remark2.5. If $\alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1, g(n) \equiv n - m + 1$, $H(m,n) \equiv (m - n)^r$. Then Corollary 2.4reduced to a special case of Theorem 1 in [7].

Theorem 2.7. Let(2.1)holds,and there exist two sequences $\xi(n)$ and $\eta(n)$ such that(2.26) and (2.27) hold.Assume that(J_1)holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$such that$x(n) > 0$ and $x(g(n)) > 0$.From the proof of Lemma 2.2$x(n)$ is either of type (I) or (II). From Theorem (2.1), $x(n)$ is not of type (I). From Theorem (2.5), $x(n)$ is not of type (II).The proof is complete.■

The proof eachof the following corollaryis similar to that of Theorem 2.7 and hence the details are omitted.

Corollary 2.5. Let(2.1)holds,and there exist two sequences $\xi(n)$ and $\eta(n)$ such that(2.26) and (2.27) hold.Assume that (J_2) holds. Then equation (1.1) is oscillatory.

Corollary 2.6. Let(2.1)holds,and there exist two sequences $\xi(n)$ and $\eta(n)$ such that(2.26) and (2.27) hold.Assume that (J_3)holds. Then equation (1.1) is oscillatory.

Corollary 2.7. Let(2.1)holds,and there exist two sequences $\xi(n)$ and $\eta(n)$ such that(2.26) and (2.27) hold.Assume that (J_4)holds. Then equation (1.1) is oscillatory.

2.4. Nonexistence of solutions of type (III)

Next, we shall establish some criteria for the nonexistence of solution of type (III) for equation (1.1).

Theorem 2.8. Assume that(S_1), $(A_1) - (A_4)$hold. If the first order delay equation

$$\sum_{s=n_0}^{\infty}\left(a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{r=n_0}^{s-1} Q(r)\, f\left(\sum_{u=n_0}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(u)\right) f\left(\sum_{v=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(v)\right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}}\right) = \infty \ , \qquad (2.28)$$

is oscillatory, then equation (1.1) has no solution of type (III).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (III). Then, there is a$n_0 \in N$such that(III) holds for$n \geq n_0$.Then, we have

$$x(n) - x(n_3) = \sum_{s=n_3}^{n-1} \Delta x(s) = \sum_{s=n_3}^{n-1} b^{-\frac{1}{\alpha_1}}(s)\left(b(s)\left(\Delta x(s)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}$$

$$\geq \left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \sum_{s=n_3}^{n-1} b^{-\frac{1}{\alpha_1}}(s)\,, for\ n \geq n_3,$$

and hence

$$x(n) \geq \left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \sum_{s=n_3}^{n-1} b^{-\frac{1}{\alpha_1}}(s)\,, for\ n \geq n_3.$$

There exists a $n_4 \geq n_3$ with $(n) \geq n_3 for\ all\ \ n \geq n_4$, such that

$$x(g(n)) \geq \left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \sum_{s=n_3}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\,, for\ n \geq n_4.$$

From equation(1.1),(S_1) and the last inequality, we obtain, for $n \geq n_4$

$$0 \geq \Delta\left(a(n)\left(\Delta v(n)\right)^{\alpha_2}\right) + Q(n) f\left(v^{\frac{1}{\alpha_1}}(g(n))\right) f\left(\sum_{s=n_3}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\right), \qquad (2.29)$$

where $v(n) := b(n)\left(\Delta x(n)\right)^{\alpha_1}$. It is clear that $v(n) > 0$ and $\Delta v(n) < 0$. It follows that
$-a(n)\left(\Delta v(n)\right)^{\alpha_2} \geq -a(n_4)\left(\Delta v(n_4)\right)^{\alpha_2} for\ n \geq n_4,$
thus

$$-\Delta v(n) \geq -\frac{a^{\frac{1}{\alpha_2}}(n_4)\Delta v(n_4)}{a^{\frac{1}{\alpha_2}}(n)} for\ n \geq n_4.$$

Summing the last inequality from n to ∞, we obtain

$$v(n) \geq -a^{\frac{1}{\alpha_2}}(n_4)\Delta v(n_4) \sum_{s=n}^{\infty} a^{-\frac{1}{\alpha_2}}(s) = K_1 \sum_{s=n}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,, for\ n \geq n_4$$

where $K_1 := -a^{\frac{1}{\alpha_2}}(n_4)\Delta v(n_4) > 0$.There exists a $n_5 \geq n_4$ with $g(n) \geq n_4 for\ all\ \ n \geq n_5$, such that

$$v(g(n)) \geq K_1 \sum_{s=g(n)}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,, for\ n \geq n_5.$$

Summing (2.29) from n_5 to $n-1$ and using the above inequality, we find

$$\sum_{r=n_5}^{n-1} Q(r) f\left(\sum_{s=n_3}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(s)\right) f\left(K_1 \sum_{k=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}} \leq a(n_5)\left(\Delta v(n_5)\right)^{\alpha_2} - a(n)\left(\Delta v(n)\right)^{\alpha_2},$$

In view of(S_1), we see that

$$\left(\frac{L}{a(n)} \sum_{r=n_5}^{n-1} Q(r) f\left(\sum_{s=n_3}^{g(r)-1} b^{-\frac{1}{\alpha_1}}(s)\right) f\left(\sum_{k=g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}} \leq -\Delta v(n),$$

where $L := f\left(K_1^{\frac{1}{\alpha_1}}\right)$.Summing the above inequality from $n_5 to\, \infty$, we obtain

$$L^{\frac{1}{\alpha_2}}\sum_{s=n_5}^{\infty}\left(a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{r=n_5}^{s-1}Q(r)f\left(\sum_{v=n_3}^{g(r)-1}b^{-\frac{1}{\alpha_1}}(v)\right)f\left(\sum_{k=g(r)}^{\infty}a^{-\frac{1}{\alpha_2}}(k)\right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}}\right) \le v(n_5) < \infty \ ,$$

which contradicts the condition (2.28). The proof is complete.■

Theorem 2.9. Assume that $(S_2), (A_1) - (A_4)$ hold. Let $\{\rho(n)\}$ be a positive sequence.If

$$\limsup_{n\to\infty}\sum_{u=n_0}^{n-1}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\sum_{s=n_0}^{u-1}a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{t=n_0}^{s-1}\Psi(t)\left(\sum_{\tau=g(t)}^{\infty}a^{-1}(\tau)\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right) = \infty . \qquad (2.30)$$

Then equation (1.1) has no solution of type (III).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (III). Then, there exists $n_2 \ge n_1$ such that $\Delta x(n) > 0$, $\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right) < 0$for all $n \ge n_2$.Then, we have

$$\Delta x(n) = \frac{\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}}{\left(b(n)\right)^{\frac{1}{\alpha_1}}}.$$

Summing the above inequality from$n_2\ \ ton\ -\ 1$, we obtain

$$x(n) - x(n_2) = \sum_{s=n_2}^{n-1}\frac{\left(b(s)\left(\Delta x(s)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}}{\left(b(s)\right)^{\frac{1}{\alpha_1}}}$$

$$\ge \left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\sum_{s=n_2}^{n-1}\frac{1}{\left(b(s)\right)^{\frac{1}{\alpha_1}}}. \ (2.31)$$

Hencethere exists a $n_3 \ge n_2$such that

$$x(g(n)) \ge \left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\sum_{s=n_2}^{g(n)-1}\frac{1}{\left(b(s)\right)^{\frac{1}{\alpha_1}}}, for\ n \ge n_3.$$

From equation (1.1), $(S_2)$and the last inequality, we obtain

$$\Delta\left(a(n)\left(\Delta v(n)\right)^{\alpha_2}\right) + KQ(n)\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)^{\alpha_2}\left(\sum_{s=n_2}^{g(n)-1}\frac{1}{\left(b(s)\right)^{\frac{1}{\alpha_1}}}\right)^{\boldsymbol{\alpha}} \le 0, n \ge n_3, \quad (2.32)$$

where $v(n) := b(n)\left(\Delta x(n)\right)^{\alpha_1}$. It is clear that $v(n) > 0$ and $\Delta v(n) < 0$. It follows that

$$\Delta\left(a(n)\left(\Delta v(n)\right)^{\alpha_2}\right) + \Psi(n)v^{\alpha_2}(g(n)) \le 0, \qquad for\ n \ge n_3. \qquad (2.33)$$

Since $g(n) \to \infty$as $n \to \infty$,wecan choose$n_4 \ge n_3$such that$g(n) \ge n_4 for\ n \ge n_4$ and thus

$$v(\infty) - v(g(n)) = \sum_{s=g(n)}^{\infty}a(s)\Delta v(s)\frac{1}{a(s)}$$

$$< \Delta v(g(n))a(g(n))\sum_{s=g(n)}^{\infty}\frac{1}{a(s)} < a(n_4)\Delta v(n_4)\sum_{s=g(n)}^{\infty}\frac{1}{a(s)}.$$

Thus

$$-v(g(n)) < a(n_4)\Delta v(n_4) \sum_{s=g(n)}^{\infty} \frac{1}{a(s)}.$$

Substituting back in (2.33), we have

$$\Delta\left(a(n)\left(\Delta v(n)\right)^{\alpha_2}\right) < L^{\alpha_2}\Psi(n)\left(\sum_{s=g(n)}^{\infty}\frac{1}{a(s)}\right)^{\alpha_2}, \qquad for\ n \geq n_4, \tag{2.34}$$

where $L = a(n_4)\Delta v(n_4) < 0$.Summing this inequality from $n_4\ \ to n - 1$, we see that

$$a(n)\left(\Delta v(n)\right)^{\alpha_2} < a(n)\left(\Delta v(n)\right)^{\alpha_2} - a(n_4)\left(\Delta v(n_4)\right)^{\alpha_2} < L^{\alpha_2}\sum_{s=n_4}^{n-1}\Psi(s)\left(\sum_{\tau=g(s)}^{\infty}\frac{1}{a(\tau)}\right)^{\alpha_2}.$$

where $\Delta v(n) < 0$.Summing again from $n_5\ \ to n - 1$, we have

$$v(n) < L\sum_{s=n_5}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{t=n_4}^{s-1}\Psi(t)\left(\sum_{\tau=g(t)}^{\infty}\frac{1}{a(\tau)}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}$$

or equivalently

$$\Delta x(n) < \left(\frac{L}{b(n)}\right)^{\frac{1}{\alpha_1}}\left(\sum_{s=n_5}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{t=n_4}^{s-1}\Psi(t)\left(\sum_{\tau=g(t)}^{\infty}\frac{1}{a(\tau)}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}.$$

Summing from $n_6\ \ to n - 1$, we have

$$x(n) < x(n_6) + L^{\frac{1}{\alpha_1}}\sum_{u=n_6}^{n-1}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\sum_{s=n_5}^{u-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{t=n_4}^{s-1}\Psi(t)\left(\sum_{\tau=g(t)}^{\infty}\frac{1}{a(\tau)}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right).$$

By condition (2.30), we have $\lim_{n\to\infty} x(n) = -\infty$ which contradicts the fact that $x(n) > 0$. The proof is complete . ■

2.5 Oscillation criteriaunder condition (2.2)

Next, we shall establish some oscillation criteria for equation (1.1)under condition (2.2).

Theorem 2.11. Let $(2.2), (2.5) and\ (2.28)$ hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0\ and\ x(g(n)) > 0$. From(2.2), there exist three possible cases(I),(II),(III). From Theorem (2.1) or (2.2) or (2.3) or (2.4) respectively, $x(n)$ is not of type (I). From Lemma (2.3), we have $\lim_{n\to\infty} x(n) = 0$.From Theorem (2.8), $x(n)$ is not of type (III). The proof is complete.■

Theorem 2.12. Let (2.2) and (2.28) hold, and there exist two sequences $\xi(n)$ and $\eta(n)$ such that (2.26) and (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$ and $x(g(n)) > 0$. Then, proceeding as in the proof of theorem (2.11), we obtain $x(n)$ is not of type (I). From Theorem (2.5), $x(n)$ is not of type (II). From Theorem (2.8), $x(n)$ is not of type (III). The proof is complete.■

Theorem 2.13. Let $(2.2), (2.5)$ and (2.30) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$ and $x(g(n)) > 0$. Then, proceeding as in the proof of theorem (2.11), we obtain $x(n)$ is not of type (I). From Lemma (2.3), we have, $\lim_{n\to\infty} x(n) = 0$. From Theorem (2.9), $x(n)$ is not of type (III). The proof is complete.■

Theorem 2.14. Let (2.2) and (2.30) hold, and there exist two sequences $\xi(n)$ and $\eta(n)$ such that (2.26) and (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory .

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(n) > 0$ and $x(g(n)) > 0$. proceeding as in the proof of theorem (2.12), we obtain $x(n)$ is not of type (I) or (II). From Theorem (2.9), $x(n)$ is not of type (III). The proof is complete.■

2.6 Nonexistence of solutions of type (IV)

Next, we shall establish some criteria for the nonexistence of solution of type (IV) for equation (1.1).

Theorem 2.15. Assume that $(S_1), (A_1) - (A_4)$ hold. If

$$\sum_{l=n_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(l)\left(\sum_{k=n_0}^{l-1} a^{-\frac{1}{\alpha_2}}(k)\left(\sum_{s=n_0}^{k-1} Q(s) f\left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right) = \infty . \qquad (2.35)$$

Then equation (1.1) has no solution of type (IV).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (IV). Then, there is a $n_0 \in N$ such that (IV) holds for $n \geq n_0$. We one can choose $n_3 \geq n_2$ with $g(n) \geq n_2$ for all $n \geq n_3$, such that

$$x(g(n)) \geq -\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)$$

$$\geq K_2 \sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r) \text{ for } n \geq n_3,$$

where $K_2 := -\left(b(g(n))\left(\Delta x(g(n))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} > 0$. Thus equation (1.1) and (S_1) yield

$$\Delta\left(a(n)\left(\Delta\left(b(n)(\Delta x(n))^{\alpha_1}\right)\right)^{\alpha_2}\right) \leq -Q(n) f\left(x(g(n))\right)$$

$$\leq LQ(n) f\left(\sum_{r=g(n)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right).$$

where $L := -f(K_2)$.Summing the above inequality from $n_3\ ton - 1$, we find

$$a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2} \le L\sum_{s=n_3}^{n-1} Q(s)f\left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right).$$

Hence,

$$\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big) \le L^{\frac{1}{\alpha_2}}\, a^{-\frac{1}{\alpha_2}}(n)\left(\sum_{s=n_3}^{n-1} Q(s)f\left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}.$$

Again summing the above inequality from $n_3\ ton - 1$, we find

$$b(n)(\Delta x(n))^{\alpha_1} \le L^{\frac{1}{\alpha_2}}\sum_{s=n_3}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{u=n_3}^{s-1} Q(u)f\left(\sum_{r=g(u)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}.$$

It follows that

$$\Delta x(n) \le \frac{1}{\alpha}K_3 b^{-\frac{1}{\alpha_1}}(n)\left(\sum_{s=n_3}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{u=n_3}^{s-1} Q(u)f\left(\sum_{r=g(u)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}.$$

where$K_3 := L^{\frac{1}{\alpha}}$.Finally, summing the above inequality from $n_3\ ton - 1$, we have

$$x(n) \le x(n_3) - K_3\sum_{s=n_3}^{n-1}\left(b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_3}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\left(\sum_{v=n_3}^{u-1} Q(v)f\left(\sum_{r=g(v)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right).$$

From condition(2.35), we get $x(n) \to -\infty$as $n \to \infty$, which contradicts the fact that $x(n)$ is a positive solution of(1.1). The proof is complete.■

Theorem 2.16. Assume that $(S_2), (A_1) - (A_4)$ hold. If

$$\sum_{u=n_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\sum_{s=n_0}^{u-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}\right) = \infty . \tag{2.36}$$

Then equation (1.1) has no solution of type (IV).

Proof. Let $x(n)$ be an eventually positive solution of equation (1.1) of type (IV). Then, there is an$n_0 \in N$such that(IV) holds for$n \ge n_0$.Since$a(n)\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)$ is non-increasing sequence there exists a negative constant $K_4\ and\ n_2 \ge n_1$ such that

$$a(n)\left(\Delta\big(b(n)(\Delta x(n))^{\alpha_1}\big)\right)^{\alpha_2} \le K_4\ for\ n \ge\ n_2.$$

Dividing by $a(n)$and summing the last inequality from $n_1 ton - 1$, we obtain

$$\Delta x(n) \le b^{-\frac{1}{\alpha_1}}(n)K_4^{\frac{1}{\alpha}}\left(\sum_{s=n_1}^{n-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}.$$

Summing the last inequality from $n_1 ton - 1$, we obtain

$$x(n) \le x(n_1) + K_4^{\frac{1}{\alpha}} \sum_{u=n_1}^{n-1} \left(b^{-\frac{1}{\alpha_1}}(u) \left(\sum_{s=n_1}^{u-1} a^{-\frac{1}{\alpha_2}}(s) \right)^{\frac{1}{\alpha_1}} \right).$$

Letting $n \to \infty$then, by (2.36) we deduce that$x(n) \to -\infty$,which is contradiction to the fact that $x(n) > 0$.

2.7.Oscillation criteriaunder condition (2.3)

Next, we shall establish some oscillation criteria for equation (1.1)under condition (2.3).

Theorem 2.17. Let $(2.3), (2.5) and\ (2.35)$ hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$holds. Then equation (1.1) is oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \ge n_0$ such that $x(n) > 0\ and\ x(g(n)) > 0$. From(2.3), there exist four possible cases(I),(II),(III)and(III). From Theorem (2.1) or (2.2) or (2.3) or (2.4) respectively, $x(n)$ is not of type (I). From Lemma (2.3), we have,$\lim_{n\to\infty} x(n) = 0$.From Theorem$(2.8) or (2.9)$respectively, $x(n)$ is not of type (III).From Theorem (2.15), $x(n)$ is not of type (IV). The proof is complete.■

Remark2.6. If $\alpha_2 \equiv 1, n \equiv 1, g(n) \equiv n - \sigma$and (J_3)holds. Then Theorem 2.17 extended and improved Theorem 15 in [15].

Theorem 2.18. Let $(2.3)\ and\ (2.35)$ hold,and there exist two sequences $\xi(n)\ and\ \eta(n)$ such that$(2.26)\ and\ (2.27)$ hold.Assume that(J_1) or (J_2) or (J_3) or (J_4)holds.And $(2.28) or (2.30)$holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \ge n_0$such that$x(n) > 0\ and\ x(g(n)) > 0$.Then, proceeding as in the proof of Theorem (2.19), we obtain $x(n)$ is not of type (I) or (III). From Theorem (2.5), $x(n)$ is not of type (II).From Theorem (2.15), $x(n)$ is not of type (IV). The proof is complete.■

Theorem 2.19. Let $(2.3), (2.5) and\ (2.36)$ hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$holds. Then equation (1.1) is oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \ge n_0$such that$x(n) > 0\ and\ x(g(n)) > 0$.Then, proceeding as in the proof of theorem (2.19), we obtain $x(n)$ is not of type (I) or (III). From Lemma (2.3), we have $\lim_{n\to\infty} x(n) = 0$.From Theorem (2.16), $x(n)$ is not of type (IV). The proof is complete.■

Remark2.7. If $\alpha_2 \equiv 1, n \equiv 1, g(n) \equiv n - \sigma$and (J_3)holds. Then Theorem 2.17 extended and improved Theorem 15 in [15].

Theorem 2.20. Let $(2.3)\ and\ (2.36)$ hold,and there exist two sequences $\xi(n)\ and\ \eta(n)$ such that$(2.26)\ and\ (2.27)$ hold.Assume that(J_1) or (J_2) or (J_3) or (J_4)holds. And $(2.28) or (2.30)$holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a nonoscillatory solution. Then, without loss of generality, there is a $n_1 \ge n_0$such that$x(n) > 0\ and\ x(g(n)) > 0$.Then, proceeding as in the proof of theorem (2.19), we obtain $x(n)$ is not of type (I) or (II) or(III). From Theorem (2.16), $x(n)$ is not of type (IV). The proof is complete.■

Remark2.8. If $b(n) \equiv 1, \alpha_2 \equiv 1, n \equiv 1$ and (J_1) hold. Then Theorem 2.20 extended and improved Theorem 2.1 in [6].

3 Examples

In this section we will show the applications of our oscillation criteria by three examples. We will see that the equations in the example is oscillatory or tend to zero based on the results in Section 2.

Example 3.1. Consider the nonlinear delay difference equation

$$\Delta\left(\frac{1}{n}\Delta^2 x_n\right) + \sum_{i=1}^{2} q_i(n)\, f\big(x(g_i(n))\big) = 0, n = 2 \qquad (3.1)$$

Allthe conditions of corollary (2.2) are satisfied (with)

$q_1(n) = 2n^6,\ q_2(n) = 4n^6, g_1(n) = \frac{n}{2}, g_2(n) = \frac{n}{3}, \rho(n) = n, \mu = K = 1.$

Hence every solution of (3.1) is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Example 3.2. Consider the linear delay difference equation

$$\Delta^3 x(n) + \left(\frac{27}{32}\right) x(n-2) = 0, n = 1. \qquad (3.2)$$

All the conditions of corollary (2.3) are satisfied (with $\rho(n) = 1, \mu = 1, H(m,n) = m - n$). Hence every solution of (3.2) is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.The sequence $\{2^{-n}\}$ is a solution of eq.(3.2).

Example 3.3. Consider the linear delay difference equation(2010-873-876)

$$\Delta\big(n\Delta^2 x(n)\big) + 4(2n+1)x^3(n-2) = 0, n = 1. \qquad (3.3)$$

By corollary (2.3),hence all solutions of (3.3) is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.In fact The sequence $\{(-1)^n\}$is one such solution of eq.(3.3).
are satisfied (with $\rho(n) = 1, \mu = 1, H(m,n) = m - n$). Hence every

Example3.4. Consider the linear delay difference equation

$$\Delta^3 x(n) + \frac{1}{n} x(n-3) = 0, n \ge 1. \qquad (3.4)$$

All the conditions of corollary (2.4) are satisfied (with$\rho(n) = n, K = (m-n)^r = 1$). Hence every solution of (3.4) is oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

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