On the orbit of a finitely generated module

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Abstract
In this paper, we will show that the orbit of a finitely generated R-module M can be expressed as the genus of M/K, where K is the kernel of a certain homomorphism. Various homomorphisms will be constructed between orbit groups.
In order to make some computations, we are going to define an equivalence relation on the group of units of $S^{-1}R$ and establish an isomorphism between the obtained quotient set and the orbit of M.
In the light of the result of this paper and following the one by Roger and Sylvia Wiegand in 1987, it could be possible to give a complete description of the genus of a finitely generated torsion-free R-module as the orbit of M under a suitable group action.

Key words: Localization, genus, orbit, stable isomorphism.

1. Introduction
Given a localization in some category C, one can introduce the concept of the genus $G(X)$ of an object X of C. We say that two objects X, Y in C belong to the same genus, or that $Y \in G(X)$, if $X_p$ is isomorphic to $Y_p$ for each prime $p$, where $X_p$ is the localization of X at a prime $p$.

One of the important question here has been the description of the genus in terms of its algebraic structure: Is it a group? Is it finite? Is it abelian? Is it computable? In many interesting cases, the genus is finite. Below are some known results from the literature:
In [2], Hilton and Mislin define an abelian group structure on the genus set $G(N)$ of a finitely generated nilpotent group N with finite commutator subgroup.
For groups in the class $N_1$, these are nilpotent groups N, given in terms of the associated short exact sequence

$$1 \to TN \to N \to FN \to 1,$$

where TN is the torsion subgroup and FN the torsion-free quotient, by the conditions

(a) $TN$ and $FN$ are commutative
(b) Relation (1.1) splits for the action $\omega : FN \to \text{Aut} TN$
(c) $\omega(FN)$ lies in the center of $\text{Aut} TN$.
(It was observed in [1] that, in the presence of (a), (c) is equivalent to (c’))
(c’) For all $\xi \in FN$, there exists an integer u such that $\xi a = \omega(\xi)(a) = ua$ for all $a \in TN$.
(Here, $TN$ is written additively.)
It is shown in [3] that the genus $G(N)$ of a group $N$ in $N_1$ is trivial unless $FN$ is cyclic. In the case $FN$ is cyclic generated by $\xi$, and $d$ is the multiplicative order of $u$ (see $(c')$) modulo $m$, where $m$ is the exponent of $TN$, then the calculation of the genus yields

$$G(N) \cong (\mathbb{Z}_d^*)^*/\{\pm 1\}$$

where $(\mathbb{Z}_d^*)$ is the multiplicative group of units in the ring $\mathbb{Z}_d$. By $\mathbb{Z}_d^*/\{\pm 1\}$ we mean the quotient group $\mathbb{Z}_d^*/\{-1, 1\}$.

For nilpotent groups which belong to class $K$ (of semidirect products of the form $T \rtimes \mathbb{Z}_k^1$, where $T$ is a finite abelian group and $k$ is a positive integer), many computations of the genus groups appear in the literature.

1.1 Definitions and notation

Let us first fix some notation and conventions that will remain in force for the rest of the paper. All rings are associative with identity, and all modules will be finitely generated modules unless explicitly stated to the contrary. Ring homomorphisms are assumed to preserve identity, and if we say that $R$ is a subring of $S$ then we assume that $R$ and $S$ have the same 1. If $M$ is a module, then $M^n$ will denote the direct sum of $n$ copies of $M$.

We recall some useful definitions.

Let $R$ be a commutative ring with a unit, and $S$ a subring of $R$.

Then $R$ is said to be reduced if $R$ contains no non-zero nilpotent elements. An element $r \in R$ is said to be integral over $S$ if there exists a monic polynomial $f$ with coefficients in $S$ such that $f(r) = 0$.

The integral closure of $S$ in $R$ is a subset of $R$ consisting of elements integral over $S$. It is a subring of $R$ containing $S$, and is denoted by $\overline{S}$. The integral closure $\overline{R}$ of a reduced ring $R$ is defined to be its integral closure in its total ring of fractions (or field of fractions for an integral domain) $Q(R)$.

The conductor of $R$ is the set $f = (R : \overline{R}) = \{a \in Q(R) \mid a\overline{R} \subseteq R\}$. It is the largest ideal of $R$ that is also an ideal of $\overline{R}$.

A ring $R$ is said to be with finite normalization if its integral closure in its quotient ring is finitely generated as an $R$-module.

Let $R$ be a one-dimensional reduced noetherian ring with finite normalization.

The localization ring $S^{-1}R$, where $S$ is the complement in $R$ of the union of the singular maximal ideals, is called the singular semilocalization of $R$.

The singular maximal ideals are those that contain the conductor $f$ of $R$.

Note that the integral closure of $S^{-1}R$ (in its total quotient ring) coincides with $S^{-1}\overline{R}$.

The localization of an $R$-module is tightly linked to the one of the ring $R$ via the tensor product $S^{-1}M = M \otimes_R S^{-1}R$. If $S$ is the complement of a prime ideal $p$, then $S^{-1}M$ is denoted by $M_p$.

For a finitely generated $R$-module $M$, we denote by $\text{genus}(M)$ the set of all isomorphism classes of finitely generated $R$-modules $N$ such that $M_m \cong N_m$ as $R_m$-modules for every maximal ideal $m$ of $R$.

A semilocal ring is the one with a finite number of maximal ideals.
1.2 Localization of a module
Let $S$ be a multiplicative subset of a commutative ring $R$ and $M$ an $R$-module.

The localization of $M$ with respect to $S$ will be denoted by $S^{-1}R \otimes_R M$. If $S$ is the complement of a prime ideal $p$, we write $M_p$ for $S^{-1}R \otimes_R M$.

The localization ring $S^{-1}R$, where $S$ is the complement in $R$ of the union of the singular maximal ideals, is called the singular semilocalization of $R$.

For a finitely generated $R$-module $M$, we denote by $\text{genus}(M)$ the set of all isomorphism classes of finitely generated $R$-modules $N$ such that $M_m \cong N_m$ as $R_m$-modules for every maximal ideal $m$ of $R$.

2. An action of the group $(S^{-1}R)^*$ on a set of finitely generated modules
Here we are going to consider an action of the group of units $(S^{-1}R)^*$ on a representative class $\text{FinGen}(R)$ of finitely generated modules $M$ over one-dimensional noetherian ring with finite normalization $R$ and describe the orbit of $M$ under this action.

The advantage here is that a given finitely generated $R$-module $M$ can be represented as a pullback:

$$
\begin{array}{ccc}
M & \rightarrow & \overline{R} \otimes_R M \\
\downarrow & & \downarrow \phi \\
S^{-1}R \otimes_R M & \mapsto & S^{-1}\overline{R} \otimes_R M
\end{array}
$$

and one can mostly work with the module $S^{-1}R \otimes_R M$ over the one-dimensional semilocal ring $S^{-1}R$.

Let $R$ be a one-dimensional noetherian reduced ring with finite normalization.

Since $S^{-1}\overline{R}$ is a finite product $D_1 \times \cdots \times D_k$ of semilocal principal ideal domains $D_i$, there is an internal decomposition of the group $(S^{-1}R)^*$.

Suppose $F$ is a finitely generated $S^{-1}\overline{R}$-module. Then $F$ has a decomposition $F_1 \times \cdots \times F_k$, where each $F_i$ is a finitely generated $D_i$-module.

Let $J$ be the set of indices $j$ ($1 \leq j \leq k$) such that the $D_j$-module $F_j$ has non-zero torsion-free rank. Given an element $\epsilon=(\epsilon_1 \times \cdots \times \epsilon_k) \in (S^{-1}R)^*$, define a new element $\epsilon|F \in (S^{-1}R)^*$ by letting the $j$th coordinate of $\epsilon|F$ be $\epsilon_j$ if $j \in J$ and 1 if $j \notin J$.

If $M$ is a finitely generated $R$-module, respectively $S^{-1}R$-module, we put $\epsilon|M:=\epsilon|S^{-1}R \otimes_R M$, respectively, $\epsilon|M:=\epsilon|S^{-1}R \otimes_{S^{-1}R} M$.

Let denote the group of units $(S^{-1}R)^*$ by $\Omega(R)$.

2.1 Construction of the group action
Let $M$ be a finitely generated $R$-module and let $\epsilon \in \Omega(R)$. Choose an arbitrary automorphism $\phi$ of $\overline{R}$.
\[ \Theta \in \text{Aut} S^{-1} R (S^{-1} R \otimes_R M) \text{ with } \det(\Theta) = \epsilon|\text{M}. \]

Define an \( R \)-module \( M^\Theta \) (eventually to be denoted by \( M^\epsilon \)) by the following pullback diagram:

\[
\begin{array}{ccc}
M^\epsilon & \rightarrow & \overline{R} \otimes_R M \\
\downarrow & & \downarrow \phi \\
S^{-1} R \otimes_R M & \rightarrow & S^{-1} \overline{R} \otimes_R M
\end{array}
\]  

The isomorphism class of \( M^\Theta \) does not depend on the choice of \( \Theta \) (see the construction [6]).

We define a correspondance

\[ \Omega(R) \times \text{FinGen}(R) \rightarrow \text{FinGen}(R) \]

by

\[ (\epsilon, [M]) \mapsto [M]^\epsilon = [M^\Theta]. \]

It follows by Proposition 3.5 in [6] that this correspondance is an action of \( \Omega(R) \) on \( \text{FinGen}(R) \).

We will denote by \( M^\epsilon \) any finitely generated \( R \)-module \( N \) for which \([N]=[M]^\epsilon\). Keeping in mind that \( M^\epsilon \) is defined only up to (non-canonical) isomorphism.

We will denote by \( Orb(M) \), the orbit of \( M \) under this action.

3. Abelian group structure on the orbit set

From Proposition 3.7 of [6], it follows that for a finitely generated \( R \)-module \( M \), its orbit \( Orb(M) \) can be described as the set of stable isomorphism classes of \( M \) and \( Orb(M) \subseteq \text{genus}(M) \).

This generalizes the result by Roger and Sylvia Wiegand [5] on the stable isomorphism class of a finitely generated faithful and torsion-free \( R \)-modules. They gave a construction of the stable isomorphism class of \( M \) as the orbit of \( M \) under an action of the group of units of \( R/f \).

From [4], the stable isomorphism class of a finitely generated \( R \)-module \( M \) can be equipped with an abelian group structure.

The orbit \( Orb(M) \) can inherited the same group structure as follows:

Let \([N],[N']\in Orb(M)\) with \( N\cong M^\epsilon \) and \( N'\cong M^\delta \) for some \( \epsilon, \delta \in \Omega(R) \).

We define the following additive operation:

\[ [N] + [N'] = [L] \quad \text{where } L \cong M^\tau \]

with \( \tau = \epsilon(\delta|M) \) or \( (\epsilon|M)\delta \).

The zero element here is the class \([M]\). Together with this operation, \( Orb(M) \) is a group. And since \( (M^\epsilon)^\delta|M\equiv (M^\delta)^\epsilon|M \), it is an abelian group.

3.1 Equivalence relation on \( Orb(M) \)

In order to make some computations of \( Orb(M) \), we defined an equivalence relation on \( \Omega(R) \) and construct an isomorphism between the set of its equivalence classes (denoted by \( \Omega(R)/(M) \)) and \( Orb(M) \). This is because its seems much easier to compute \( \Omega(R)/(M) \) than \( Orb(M) \).

For a finitely generated \( R \)-module \( M \), we have the following map

\[ \varphi_M : \Omega(R) \rightarrow Orb(M) \]

defined by \( \varphi \mapsto M^\varphi \) which, unfortunately is not well-defined.
To overcome this difficulty, in \( \Omega(R) \), we define an equivalence relation as follows:

For \( \varepsilon, \delta \in \Omega(R) \), \( \varepsilon \sim \delta \) if and only if \( M^\varepsilon \cong M^\delta \). We can easily check that this is an equivalence relation.

We denote by \([\varepsilon]\), the equivalence class of \( \varepsilon \) and by \( \Omega(R)/\langle M \rangle \) the set of all equivalence classes of \( \Omega(R) \) under this relation.

### 3.2 Group structure on \( \Omega(R)/(M) \)

Multiplication in \( \Omega(R)/(M) \) can be defined by \([\varepsilon][\delta] = [\varepsilon \delta]\).

The identity class element consists of all elements of the *isotropy group* of the previous group action.

**Proposition 3.2.1** The map \( \psi: \Omega(R)/(M) \rightarrow \text{Orb}(M) \) defined by \( \psi([\varepsilon]) = [M^\varepsilon] \) is well-defined and it is an isomorphism.

**Proof**

Let \( \varepsilon, \delta \in \Omega(R) \).

\[
\psi([\varepsilon \delta]) = [M^\varepsilon \delta] = [M^\varepsilon] + [M^\delta] = \psi([\varepsilon]) + \psi([\delta]).
\]

Thus \( \psi \) is a homomorphism and, from \((*)\), it is surjective.

Moreover, \( \ker \psi = \{ [\varepsilon] | \psi([\varepsilon]) = [M] \} \)

which is equal to \( \{ [\varepsilon] | M^\varepsilon \cong M \} = \{ [1] \} \)

Thus \( \psi \) is a monomorphism and the result follows.

This theorem tells us that for a one-dimensional reduced noetherian ring \( R \) with finite normalization, the problem of finding the orbit class group of an \( R \)-module \( M \) can be reduced to that of finding the quotient set of the group of units of the integral closure of the semilocal ring \( S^{-1}R \) under a suitable equivalence relation.

### 4. Homomorphism between \( M \) and \( M^\Theta \)

In what follows, for a given \( \Theta \), we construct a module homomorphism \( \varphi_\Theta \) between \( M \) and \( M^\Theta \). If \( N \) is a submodule of \( M \) that contains the kernel of \( \varphi_\Theta \), a function can be defined between orbits of \( M \) and \( M/N \).

Given the standard pullback of \( M \),

\[
\begin{array}{ccc}
  M & \rightarrow & \overline{R} \otimes_R M \\
  \downarrow & & \downarrow \varphi \\
  S^{-1}R \otimes_R M & \rightarrow & S^{-1}\overline{R} \otimes_R M
\end{array}
\]

Let \( \Gamma \in \text{Aut}_R M \) be an involution. Let \( \varphi = \eta \otimes \Gamma \) and \( \psi = \nu \otimes \text{id}_M \) where \( \eta: \overline{R} \rightarrow S^{-1}\overline{R} \) and \( \nu: S^{-1}R \rightarrow S^{-1}\overline{R} \) an inclusion map. As a pullback

\[
M = \{(r \otimes m, q \otimes m') \in (\overline{R} \otimes_R M) \times (S^{-1}M) \text{ such that } \varphi(r \otimes m) = \psi(q \otimes m') \}
\]

Let \( \varepsilon \in \Omega(R) \) and \( \Theta = \gamma \otimes \Gamma \) an automorphism of \( S^{-1}R \otimes_R M \) with \( \det(\Theta) = \varepsilon|M \).

We define a map

\[
\varphi_\Theta: M \rightarrow M^\Theta
\]

as follows:

\[
(r \otimes m, q \otimes m') \mapsto (r \otimes m, \gamma^{-1}(q) \otimes \Gamma(m'))
\]
Proposition 4.1

Let $\Gamma \in \text{Aut}_R M$ be an involution and $\gamma$ an automorphism of $S^{-1} R$. Let $\Theta = \gamma \otimes \Gamma$ be an automorphism of $S^{-1} R \otimes R M$. Then $\phi_{\Theta}$ is a surjective homomorphism of modules.

Proof.

$\phi_{\Theta}$ is well-defined:

Let $(r \otimes m, q \otimes m') \in (\overline{R} \otimes R M) \times (S^{-1} M)$. We want to show that $\varphi_{\Theta}(r \otimes m, q \otimes m') \in M^{\Theta}$.

$$
\Theta \psi(\gamma^{-1}(q) \otimes \Gamma(m')) = \Theta (\nu(\gamma^{-1}(q)) \otimes \Gamma^2(m'))
= \gamma \nu(\gamma^{-1}(q)) \otimes \Gamma^2(m')
= q \otimes m'
= \nu(q) \otimes m'
= \eta(r) \otimes \Gamma(m)
= \varphi(r \otimes m)
$$

$\phi_{\Theta}$ is an $R$-homomorphism since $\gamma^{-1}$ and $\Gamma$ are.

For $s \otimes e \in S^{-1} R \otimes M$ there exists $q \otimes m \in S^{-1} R \otimes M$ with $\nu(q) = \gamma(\nu(s))$ and $m = \Gamma^{-1}(m')$.

Thus $\phi_{\Theta}$ is a surjective homomorphism.

4.1 Orbit of a quotient module

We shall keep in mind that for $e \in \Omega(R)$, $[M]^e = [M^\Theta]$, where $\Theta$ is an automorphism of $S^{-1} \overline{R} \otimes M$ and $\det \Theta = e | M$. Let $\phi_{\Theta}: M \rightarrow M^\Theta$ be the homomorphism defined previously. Let $K_{\Theta} = \text{Ker} \phi_{\Theta}$. Then by the first isomorphism theorem

$$
L = M / K_{\Theta} \cong M^\Theta
$$

since $\phi_{\Theta}$ is surjective.

Let $N$ be a submodule of $M$ with $K_{\Theta} \subseteq N$ and let $T = N / K_{\Theta}$. By the third isomorphism theorem, $L / T \cong M / N$.

Let $T = h(T)$ where $h$ is the isomorphism between $L$ and $M^\Theta$. By short-five lemma argument, it follows that $M / N \cong M^\Theta / T$. We have the following diagram

$$
\begin{array}{ccc}
M / K_{\Theta} & \xrightarrow{z} & M^\Theta \\
\downarrow & & \downarrow \\
M / N & \xrightarrow{z} & M^\Theta / T
\end{array}
$$

We can therefore define a map

$\rho : \text{Orb}(M) \rightarrow \text{Orb}(M / N)$

in such a way that $K \mapsto K / T'$. 

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4.2 Orbit of a direct sum with a fixed module

Let $M, N$ be two finitely generated $R$-modules with $M \oplus M \in \text{genus}(N)$ and consider a map

$$ \phi_{M,N} : \text{Orb}(M) \to \text{Orb}(N) $$

We say that $\phi_{M,N}$ is defined if, for each $L \in \text{Orb}(M)$ there exists $K \in \text{Orb}(N)$, unique up to isomorphism, such that $L \oplus N \cong M \oplus K$. This is an analog of a function defined between genus class groups.

**Proposition 4.2.1** If $\phi_{M,N}$ is defined, then it is a group homomorphism.

**Proof.** Let $L, L'$ be in $\text{Orb}(M)$ and $K \in \text{Orb}(N)$ such that $[L]+[L']=[K]$, i.e. $L \oplus L' \cong M \oplus K$. Let $[F]=\phi_{M,N}([L])$, $[F']=\phi_{M,N}([L'])$ and $[X]=\phi_{M,N}([K])$. We want to prove that $[F]+[F']=[X]$, i.e., $F \oplus F' \cong N \oplus X$.

By definition of $\phi_{M,N}$,

1. $L \oplus N \cong M \oplus F$
2. $L' \oplus N \cong M \oplus F'$
3. $K \oplus N \cong M \oplus X$

(1)$\oplus$(2) gives

$$ L \oplus N \oplus L' \oplus N \cong M \oplus F \oplus M \oplus F' $$

and since

$$ L \oplus L' \cong M \oplus K, $$

we have

$$ M \oplus K \oplus N \oplus N \cong M \oplus M \oplus F \oplus F' $$

hence

$$ M \oplus M \oplus X \oplus N \cong M \oplus M \oplus F \oplus F' $$

by relation (3).

Thus $X \oplus N \cong F \oplus F'$ since $M \oplus M \in \text{genus}(N)$ and direct-sum cancellation holds in $\text{genus}(N)$.

We can now define the ‘crossing map’

$$ \phi_{M \oplus M, N} : \text{Orb}(M) \to \text{Orb}(M \oplus N) \quad \text{for which for any } X \in \text{Orb}(M), $$

$$ \phi_{M \oplus M, N}(X) = [X \oplus N] \quad . $$
References