Arbitrary Order Iterations

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Abstract
In this work, we consider the following iteration method

$$x_{n+1} = \sum_{i=0}^{r} x_n (1 - p x_n)^i$$

We will show that it converges to 1/p under the assumption $|1 - px_0| < 1$. *We also show that the order of convergence is r+1. This result is a generalization of the following iteration*

$$x_{n+1} = x_n[px_n(px_n - 3) + 3], n = 0, 1, \ldots$$

which converges to 1/p and has order of convergence equal to three [1]

Keywords: Iteration, division-free, Order of Convergence.

1. Introduction

In [1], the author used a simple method to generate second and third order iterative division-free formulas. In this note we provide an arbitrary order iteration division-free formula for which the second and third order iterations are particular cases.

2. Main Result

We consider the following iteration method which is a division-free formula

$$x_{n+1} = \sum_{i=0}^{r} x_n (1 - p x_n)^i$$

We will show that it converges to 1/p under the assumption $|1 - px_0| < 1$. We also show that the order of convergence is r + 1. First, we will prove the following lemmas.

Lemma 2.1

$$1 - px_n - px_n(1 - px_n) - px_n(1 - px_n)^2 - \cdots - px_n(1 - px_n)^r = (1 - px_n)^{r+1}$$

Proof:Consider first $r = 1$.

$$\begin{aligned} 1 - px_n - px_n(1 - px_n) &= 1 - 2px_n + p^2x_n^2 \\ &= (1 - px_n)^2 \end{aligned}$$

Suppose the result is correct up to order r. Let us prove it for order $r + 1$.

$$\begin{aligned} 1 - px_n - \cdots - px_n(1 - px_n)^{r+1} &= (1 - px_n)^{r+1} - px_n(1 - px_n)^{r+1} \\ &= (1 - px_n)^{r+1}(1 - px_n) \\ &= (1 - px_n)^{r+2} \end{aligned}$$

Lemma 2.2

$$1 - px_{n+1} = (1 - px_n)^{r+1}$$

Proof: x_{n+1} is given by the following expression.

$$x_{n+1} = \sum_{i=0}^{r} x_n(1 - px_n)^i$$

Therefore

$$\begin{aligned} px_{n+1} &= \sum_{i=0}^{r} px_n(1 - px_n)^i \\ &= px_n + px_n(1 - px_n) + \cdots + px_n(1 - px_n)^r \end{aligned}$$

Hence

$$\begin{aligned} 1 - px_{n+1} &= 1 - px_n - px_n(1 - px_n) - \cdots - px_n(1 - px_n)^r \\ &= (1 - px_n)^{r+1} \end{aligned}$$

by Lemma 2.1.

Lemma 2.3

$$(1-px_n)=(1-px_0)^{(r+1)^n}$$

Proof: By mathematical induction. For $n=0$, it's true clearly. Suppose the result is true for n. Let us prove it for $n+1$.

$$\begin{aligned}1-px_{n+1} &= 1-px_n-px_n(1-px_n)-\cdots-px_n(1-px_n)^r\\ &= (1-px_n)^{r+1} \qquad Lemma\ \ 2.1\\ &= [(1-px_0)^{(r+1)^n}]^{r+1}\\ &= (1-px_0)^{(r+1)^{n+1}}\end{aligned}$$

Lemma 2.4

$$x_n=x_0+x_0(1-px_0)+x_0(1-px_0)^2+\cdots+x_0(1-px_0)^{(r+1)^n-1}$$

Proof: By mathematical induction. For $n=0$, it's obvious. Suppose the result is true for n. Let us prove it for $n+1$.

$$\begin{aligned}x_{n+1} &= x_n+x_n(1-px_n)+x_n(1-px_n)^2\cdots+x_n(1-px_n)^r\\ &= x_n+x_n(1-px_0)^{(r+1)^n}+x_n(1-px_0)^{2(r+1)^n}\\ &\quad \cdots+x_n(1-px_0)^{r(r+1)^n} \qquad Lemma\ \ 2.3\\ &= x_0+x_0(1-px_0)+x_0(1-px_0)^2+\cdots+x_0(1-px_0)^{(r+1)^n-1}\\ &\quad +x_0(1-px_0)^{(r+1)^n}+x_0(1-px_0)^{(r+1)^n+1}\\ &\quad +\cdots+x_0(1-px_0)^{2(r+1)^n-1}\\ &\quad +x_0(1-px_0)^{2(r+1)^n}+x_0(1-px_0)^{2(r+1)^n+1}\\ &\quad +\cdots+x_0(1-px_0)^{3(r+1)^n-1}\\ &\quad +x_0(1-px_0)^{r(r+1)^n}+x_0(1-px_0)^{r(r+1)^n+1}+\\ &\quad +\cdots+x_0(1-px_0)^{r(r+1)^n+(r+1)^n-1}\end{aligned}$$

Theorem 2.1 *The following iteration method*

$$x_{n+1} = \sum_{i=0}^{r} x_n(1-px_n)^i, \qquad n = 0, 1, ...$$

converges to $1/p$ *under the assumption* $|1-px_0| < 1$.

Proof:By Lemma 2.4, we have

$$x_{n+1} = x_0 + x_0(1-px_0) + x_0(1-px_0)^2 + \cdots + x_0(1-px_0)^{(r+1)^{n+1}-1}$$

Therefore

$$x_{n+1} = x_0(1 + (1-px_0) + (1-px_0)^2 + \cdots + (1-px_0)^{(r+1)^{n+1}-1})$$

Thus, as $n \to \infty$

$$x_{n+1} \to x_0/(1-(1-px_0)) = 1/p$$

Recall the definition of order of convergence which is given in most numerical analysis books [2].

Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|x_{n+1}-p|}{|x_n-p|^{\alpha}} = \lambda$$

Theorem 2.2 *The order of convergence of our iteration method is* $r+1$

Proof: By Lemma 2.2

$$1 - px_{n+1} = (1-px_n)^{r+1}$$

Hence,

$$p(1/p - x_{n+1}) = [p(1/p - x_n)]^{r+1}$$

Therefore

$$|1/p - x_{n+1}| = |p^r||1/p - x_n|^{r+1}$$

3. Concluding Remarks

We presented in this note a division-free iteration method of arbitrary order. Our future work is to apply this method to investigate the computation of generalized inverses as in [3]

References

[1] Joseph J. Roseman and G. Zwas, ”Second Order Iterations,” The College Mathematics Journal, vol. 30, No .5, November 1999.

[2] R. L. Burden and J. D. Faires, ”Numerical Analysis,” 9th edition.

[3] T. Soderstrom and G. W. Stewart, ”On The Numerical Properties Of An Iterative Method For Computing The Moore-Penrose Generalized Inverse,” SIAM J. Numer. Anal. , vol. 11, No. 1, March 1974.