One Possible Optimization of Economic design for $\overline{X}$ Chart

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Abstract

Control charts are the most widely used tool for statistical control of production processes because of their simplicity. Control chart should be designed economically in order to achieve minimum quality control costs. In the order to use any chart, it is necessary to economically define the parameters (n, k, h) where n is the sample size, h is the sampling frequency or interval between the samples and k is the width of the control limits. Because of that, the objective of this study is to create models which can be used for determining the parameters (n, k, h) for different values of the economical variables that rely on the objective function by:

1 -*Determination of objective function (function of loss by unit of the product) that connects the production process and its characteristics with the $\overline{X}$ chart, its attributes and the economical consequences of its usage.*

2- Defining the values of parameters (n, k, h) which create the objective function as small as possible for different values of the economical incomes.

3 -Use of the results obtained in previous step in order to create mathematical methods which will be used to determine the values of the parameters depending the values of reliable economical variables.

Key-words: control chart, $\bar{X}$ chat, economical design, optimization, modeling, algorithms.

1. Introduction

Statistical process control is an effective method for improving a firm's quality and productivity. There is an increased interest in their effective implementation in industry became of increase competition and improvements in quality in foreign-made products. Many tools may be utilized to gain the desired information on a firm's quality and productivity. Some of the more commonly used tools are control charts, which are useful in determining any changes in process performance.

Economical design of the $\overline{X}$ chart is also a rich area for research. A great deal of research was conducted in this area even though most of it did not give models (mathematical equation), which could be used in calculating the values of parameters for different economical variables. Duncan (in1965) proposed an economic model for the optimum economic design of $\overline{X}$ control chart. His paper was first to deal with a fully economic model of Shewhart-type control chart by incorporate formal optimization methodology into determining the control chart parameters. Duncan's paper was the stimulus for much of the subsequent research in this area. the model presented by Duncan. Subsequently, there were many works in this subject. Duncan's model from 1956 was used in 1968, by Goel, A.L and Jain, S.C and WU, S.M in order to find an algorithm for the determination of the economic design of X-charts. Feet Doglas (in 1980) gave an analysis of the optimal economic model. Van Colloni (in 1980) suggested and developed a simplified method to solve the problem of the economic design of the $\overline{X}$ chart. saniga (In 1989), develop method for the economic design of chart, taking into account the statistical properties of the economic statistical design. Deeb, Ali M. and Hamoda, Mona M. (in 2006) gave models for the optimal economic design for 3σ + $\overline{X}$ chart with warning limits. In this paper we will found models for optimal economic design of $k\sigma$ + $\overline{X}$ chart without warning limit.

To formulate an economic model to design a control chart, it is necessary to make certain assumptions. The assumptions summarized below are relatively standard in most economic models. The assumptions deal with process behaviors, statistical properties of the chart, control procedure, and economic factors.

1.1-The process behavior and assumptions

The process under investigation is assumed to have following characteristics by:

I. Production has mean of υ items per hour.

II. The quality Characteristic of interest X has a normal distribution with the mean μ and variance σ^2 known.

III. The process is considered as a series of cycles. Each cycle starts in state I under control (in control state) with mean μ_0 (target level) and ends with state II (out control state) with $\mu \neq \mu_0$. Upon the detection and correction of the assignable cause, the process returns to state 1 again and new cycle begins and so on. It is subject to one assignable cause which causes a shifts the of know magnitude ($\pm\delta\sigma$) in the process mean. The occurrence of the assignable cause shifts the process to the state II with mean $\mu_1 = \mu_0 - \delta\sigma$ with probability $P(\mu = \mu_1)$ or $\mu_2 = \mu_0 + \delta\sigma$ with probability $P(\mu = \mu_2)$. Here δ is a known shift size, and σ^2 is known and remains unchanged.

IV. Time (T) within the process that is in-control is considered to be a random variable and has an exponential distribution of a parameter λ. It means that the average duration of the state which is under control in the cycle is: E (T) = $\frac{1}{\lambda}$

V. The states of the process can be recognized by charting only.

In our analysis we will calculate new quantities based on I – IV and we give here some notation and abreviations:

1. ARL1 is the average run length when the process is in control.
2. ARL2 is the average run length when the process out of control
3. μ_s is the average number of sample in a cycle, $\mu_s = \frac{1}{(e^{\lambda h}-1)} + ARL2$
4. μ_F is the average number of a false alarm, $\mu_F = \frac{1}{(e^{\lambda h}-1)ARL1}$
5. The costs considered are:

- c_1^* is the fixed cost of a sample.
- c_2^* is the cost per unit sampled.
- c_3^* is the cost of a false alarm.
- $(c_1^* + c_2^* n)\mu_s$ is the cost of sampling in the cycle.

- c_4^* is the cost of the renewal action.
- $c_3^*\mu_F$ is the average cost per false alarm investigation.
- b^* is the expected number net benefit per renewal.
- $\mu_F h\upsilon$ is the average number of units produced per cycle.

We also will we use the following relative quantities: the relative fixed cost per sample is $C_1 = \frac{c_1^*}{c_3^*} \geq 0$, the relative cost per unit sampled is $C_2 = \frac{c_2^*}{c_3^*} \geq 0$ and the relative benefit renewal is $b_2 = \frac{b^*}{c_3^*} \geq 0$

2. Objective function

The objective function $L(n,k,h)$ is constructed by combining some of these characteristics of the production process:

$$L(n,k,h) = \frac{c_3^*}{h\upsilon}\left[c_1 + c_2 n - \frac{b-\mu_F}{\mu_S}\right]$$

Then

$$L(n,k,h) = \frac{c_3^*}{h\upsilon}\left\{c_1 + c_2 n - \frac{b\left(e^{\lambda h}-1\right) - \frac{1}{ARL1}}{1 + ARL2\left(e^{\lambda h}-1\right)}\right\}$$

The value (n^*,k^*,h^*) are the optimum design for the $\overline{X}$ control chart iff

$L(n^*,k^*,h^*) \leq L(n,k,h), \forall(n,k,h)$

In order to reduce the number of the parameters of the objective function and also to simplify the optimization procedure, we will multiply it by $\frac{\upsilon}{\lambda c_3^*}$, and let denote $s = \delta\sqrt{n}$ $x = \lambda h$ and $c = \frac{c_2}{\delta^2}$ then the standardized loss function is given by:

$$L_S(s,k,x) = \frac{1}{x}\left[c_1 + cs^2 - \frac{b\left(e^{x}-1\right) - \frac{1}{ARL1}}{1 + ARL2\left(e^{x}-1\right)}\right]$$

Then, the economic design $\overline{X}$ chart with parameters s^*,k^*,x^* is called an optimal iff

$L_s(s^*,k^*,x^*) \leq L_S(s,k,x), \forall(s,k,x)$.

2.1- Numerical minimization

The minimizing problem of the objective function $L_s(s,k,x)$, was solved by using MATLAB Optimization: Toolbox for minimization of nonlinear function with constraints for $c_1 = 0$, c in the interval $[0.0001,1]$, and b in the interval $[10,860]$. A sample of the output of this program (s^*,k^*,x^*) given in table (2.1).

Table (2.1) sample of output (s^*,k^*,x^*) and loss function for given values of b,c

b	c	s^*	k^*	x^*	L_S
10	0.0401	2.3197	1.5193	0.2441	-7.1111
60	0.0606	2.1796	1.3918	0.1047	-51.3540

110	0.0591	2.2194	1.4248	0.0758	-98.1887
160	0.0701	2.1127	1.3423	0.0654	-144.6392
210	0.0321	2.6057	1.2742	0.0589	-196.6382
260	0.0801	2.0335	1.2784	0.0527	-239.7041
310	.01910	2.8967	1.9686	0.0313	-296.5662
360	0.0371	2.5303	1.6744	0.0360	-341.4782
410	0.0301	2.6526	1.7719	0.0316	-391.6524
460	0.0291	2.6730	1.7881	0.0295	-440.7895
510	0.0471	2.3925	1.5637	0.0324	-485.9551
560	0.0291	2.6751	1.7895	0.0267	-538.7770
610	0.0491	2.3694	1.5450	0.0299	-583.2849
660	0.0551	2.2985	1.4885	0.0297	-631.0765
710	0.0471	2.3963	1.6664	0.0274	-681.5591
760	0.0711	2.1350	1.3583	0.0296	-726.1856
810	0.0651	2.1936	1.4056	0.0280	-776.0662
860	0.0621	2.2253	1.4306	0.0268	-825.5621

2.1.1-modeling

A non-linear regression models are built through the use of a package (Sigma plot) on the results obtained from the standardized loss function (s^*, k^*, x^*). They are depend variables while (c_1, c, b) as independent variables. In this case when $c \geq 0.067$ and $(0.105 - c)\sqrt{b} \leq 0.12638$ we obtain the $s^* = k^* = x^* = 0$ following models:

In all cases we obtain that $R^2 = 0.9999$ and P-value were for all parameters and coefficients less than the 0.0001.

1. Model (s^*)

$$\hat{s}^* = \begin{cases} \dfrac{1.2980b^2 - 1.3342\ln(b^2) - 115090.6604c^2/\ln(b^2) - 40756.0351c^2b^2 + 0.0091c\ln(b)}{b^2 + 1323037.3265c^4/b^2 - 22616.3265c^2b^2 + 790.8770cb^2 - 1.7369\ln(b^2)} - \dfrac{6.5210c}{\ln(b)} + 3.4296 + 0.0091\ln(b) - 17.3746c^2 \\ 0 \end{cases}$$

2. Model (k^*)

$$\hat{k}^* = \begin{cases} \dfrac{1.8569b + 2.1325c^c - 0.0177\ln(b) + 8.4195cb}{b(70486c^c - 58.8482 + 51.4296c)} - \dfrac{8.6777}{\ln(b)} \\ 0 \end{cases}$$

3. Model (x^*)

$$\hat{x}^* = \begin{cases} \dfrac{0.0001b + 25.2567bc^2 - 61.3759c^3b + 264477c\ln(b) - 0.0366\ln(b) - 1699cb + 0.1659}{\ln(b) + 1.2856c^2b^2 - 0.0190cb^2 + 24.9502cb} + \dfrac{77996.3927}{b^4} + 0.0217c\ln(b) \\ 0 \end{cases}$$

The following table contains the estimated values $\left(\hat{s}^*, \hat{k}^*, \hat{x}^*\right)$ obtained by using the previous models for some values (c, b) :

Table (2.2) of estimated normative values $\left(\hat{s}^*, \hat{k}^*, \hat{x}^*\right)$

b	c	$\hat{s}^*$	$\hat{k}^*$	$\hat{x}^*$	$\hat{L}_s$
10	0.0401	2.3190	1.5294	0.2415	-7.1111*
60	0.0601	2.1867	1.3988	0.1047	-51.3539*
110	0.0591	2.2164	1.4262	0.0761	-98.1886
160	0.0701	2.1128	1.3372	0.0658	-144.8491
210	0.0321	2.6021	1.7316	0.0448	-196.6374
260	0.0801	2.0325	1.2706	0.0527	-239.7035
310	0.0191	2.8966	1.9599	0.0305	-296.5597
360	0.0371	2.5267	1.6724	0.0359	-341.4781
410	0.0301	2.6494	1.0661	0.0314	-391.6520
460	0.0291	2.6703	1.7815	0.0294	-440.7891
510	0.0471	2.3907	1.5691	0.0324	-485.9547
560	0.0291	2.6730	1.7830	0.0268	-538.7767
610	0.0491	2.3686	1.5511	0.0299	-583.2845
660	0.0551	2.2994	1.4947	0.0296	-631.0763
710	0.0471	2.3962	1.5726	0.0274	-681.5587
760	0.0711	2.1389	1.3586	0.0295	-726.1854
810	0.0651	2.2918	1.4879	0.0268	-777.7131
860	0.0621	2.2291	1.4351	0.0268	-825.5619

Notice that, by comparing the results in the (Table 2.1) with those in the (Table 2.2) it is obvious that there are no significant differences between actual values $\left(s^*, k^*, x^*\right)$ and estimated values $\left(\hat{s}^*, \hat{k}^*, \hat{x}^*\right)$. However, some differences have been found, which have a slight impact on the standardized loss function.

2.1.*2*- Algorithms

The next will be used to algorithms to obtain the approximate optimal sampling interval (h^*), the approximate optimal control limits (k^*), as well as the approximate optimal sample size (n^*) using the modeling of the previous section, the economic consequences and the process characteristics.
The following values have been given:

1. the mean of in-control period $\frac{1}{\lambda}$
2. the cost parameter:

- b^* benefit per renewal
- c_1^* fixed cost per sample (in this study $c_1^* = 0$)
- c_2^* cost per unit sampled
- c_3^* cost per erroneous inspection

3. Shift parameter δ
4. calculate relative economical consequences

$b = \frac{b^*}{c^*_3}$, $c = \frac{c^*_2}{c^*_3\delta}$ will be used as known.

1- $\hat{h}^*$ Algorithm

To determine the approximate optimal sample interval $\hat{x}^*$ determine the following:

- Determine the optimal values of $\hat{x}^*$ from x^*model at the specified values of the relative economic consequence using 1-4 above.
- The optimal value ($\hat{h}^*$ the approximate samples interval) is obtained through the relationship $\hat{h}^* = \frac{\hat{x}^*}{\lambda}$

  The results of applying this algorithm for selected values b, c and λ is given in the table

Table (2.3) presents some approximate values optimal given of control limits

b	c	h^*			$\hat{h}^*$		
		$\lambda = 0.05$	$\lambda = 0.05$	$\lambda = 3$	$\lambda = 0.05$	$\lambda = 0.05$	$\lambda = 3$
10	0.0411	4.924	0.1641	0.0821	4.874	0.1625	0.0812
60	0.0391	1.840	0.0613	0.0307	1.846	0.0615	0.0376
110	0.0711	1.596	0.0532	0.0266	1.606	0.0535	0.02677
160	0.0501	1.192	0.0397	0.0199	1.194	0.0398	0.0199
210	0.0351	0.932	0.0311	0.0155	0.924	0.0308	0.0154
260	0.0461	0.906	0.0302	0.0151	0.908	0.0303	0.0151
310	0.0241	0.676	0.0225	0.0113	0.666	0.0222	0.011
360	0.0401	0.736	0.0245	0.0123	0.738	0.0246	0.0123
410	0.0441	0.710	0.0237	0.0118	0.710	0.237	0.0118
460	0.0391	0.606	0.0215	0.0108	0.646	0.0215	0.0108
510	0.0321	0.576	0.0192	0.0996	0.578	0.0193	0.096
560	0.0601	0.662	0.0221	0.0110	0.658	0.0219	0.011
610	0.0311	0.522	0.0174	0.0087	0.526	0.0175	0.088
660	0.0101	0. 342	0.0114	0.0057	0.304	0.0113	0.0057
710	0.0471	0.550	0.0183	0.0092	0.550	0.0183	0.0092
760	0.0611	0.568	0.0189	0.0095	0.568	0.0189	0.0095
810	0.0391	0.484	0.0161	0.0081	0.488	0.0163	0.0081
860	0.0511	0.508	0.0169	0.0085	0.508	0.0169	0.0085

2- $\hat{k}^*$ Algorithm

The optimal values of control limits $\hat{k}^*$ is obtained directly from k^* model at the values of b and c specified using 1-4 above.

The following table (2.4) presents the result of applying $\hat{k}^*$ algorithm for selected value of b and c.

Table (2.4) shows the estimated optimal values ($\hat{k}^*$)

b	c	k^*	$\hat{k}^*$
10	0.0411	1.5062	1.5163
60	0.0391	1.6187	1.6212
110	0.0711	1.3252	1.3186

160	0.0501	1.5177	1.5220
210	0.0351	1.6942	1.6925
260	0.0461	1.5665	1.5708
310	0.0241	1.8684	1.8592
360	0.0401	1.6377	1.6395
410	0.0441	1.5935	1.5975
460	0.0391	1.6521	1.6524
510	0.0321	1.7444	1.7403
560	0.0071	2.3626	2.3830
610	0.0311	1.7601	1.7553
660	0.0101	2.2300	2.2398
710	0.0471	1.5663	1.5626
760	0.0611	1.4374	1.4423
810	0.0391	1.6565	1.6586
860	0.0511	1.5282	1.5354

3- $\hat{n}^*$ Algorithm

To determine the approximate optimal sample size $\hat{n}^*$, first the optimal value of $\hat{s}^*$ is obtained from s^* model at the values of b and c specified using 1-4 above.

Then the approximate optimal sample size $\hat{n}^*$ is given by:

$$\hat{n}^* = \left[\frac{(\hat{s}^*)^2}{\delta^2} \right]$$

The result of applying this algorithm for selected values of b, c and δ is presented in the following table:

Table (2.5) of values for the optimal sample size ($n^*, \hat{n}^*$) upon determined values for each of the (b, c, δ)

b	c	n^*			$\hat{n}^*$		
		$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$	$\delta = 0.5$	$\delta = 1.5$	$\delta = 2$
10	0.0411	21	2	1	21	2	1
60	0.0391	24	3	2	24	3	2
110	0.0711	17	2	1	18	2	0
160	0.0501	22	2	1	22	2	1
210	0.0351	26	3	2	26	3	2
260	0.0461	23	3	1	23	3	1
310	0.0241	31	3	2	31	3	2
360	0.0401	25	3	2	25	3	2
410	0.0441	24	3	1	24	3	1
460	0.0391	25	3	2	25	3	2
510	0.0321	27	3	2	27	3	2
560	0.0601	20	2	1	20	2	1
610	0.0311	28	3	2	28	3	2
660	0.0101	41	5	3	42	5	3
710	0.0471	23	3	1	23	3	1
760	0.0611	20	2	1	20	2	1
810	0.0391	25	3	3	25	3	2
860	0.0511	22	2	1	22	2	1

***3.* Results**

As if can be seen from our results there are different effect on economical consequences for the optimal design of $\overline{\mathrm{X}}$ - chart

- Increasing the per unit (relative) sampling cost (the relative c_2) c_2^* leads to decrease the optimum sample size $(\hat{n}^*)$ and optimum control limits $(\hat{k}^*)$, but increase the optimum interval between samples $(\hat{h}^*)$.

- Increasing the benefit per renewal (the relative b) b^* leads to increase both the optimum sample size $(\hat{n}^*)$ and optimum control limits $(\hat{k}^*)$, but decreases the optimum interval between samples $(\hat{h}^*)$.
- Increasing the intensity parameter (λ) decreases the optimum interval between samples $(\hat{h}^*)$.

- Increasing of the (δ) decreases optimum sample size $(\hat{n}^*)$.

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