

# Pyramidization of Polygonal Prisms and Frustums

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## Introduction

I have taught a geometry course for teachers at a Christian college and a State college in the past few years. Topics covered in this course include measurement, transformation geometry, plane and space figures, problem solving in geometry, methods and materials for teaching geometry [7].

The relationship between the formulas for volumes of solids has always been difficult for some students who took the above course. For example, it was difficult for them to see that the volume of a pyramid is one third of the volume of a prism with equal base and height. They had to either memorize this relationship or be able to see this relationship by actually visualizing three pyramids fitting in the prism of the same base and height. The visualization was difficult for them, and I designed a series of methods for dividing prisms into pyramids [9] to help them see the relationship between their volumes. Then we extended our division methods to truncated square pyramids. The Geometer's Sketchpad was used in these exercises.

In this paper I describe the various sectioning methods we used to see the relationship between the volumes of polygonal pyramids with volumes of polygonal prisms and truncated pyramids. The students realized that prisms and truncated pyramids can be divided into polygonal pyramids; the same way that polygons can be divided into triangles. Pyramids are made of triangular lateral faces and a polygonal base that can be divided into triangles. Therefore, pyramids play the same role in three-dimensional geometry that triangles do in the plane. These methods present practical exercises for better understanding polygonal space figures. Specifically, students divided some polygonal prisms and a truncated square pyramid into polygonal pyramids and proved that the volumes of the resulting pyramids add up to the volume of the original solids, a tedious but worthwhile exercise. Showing that the sum of the volumes of the resulting pyramids equals the volume of the original solid requires algebraic and sometimes trigonometric computations. The students were faced with problem-solving situations that they had created by dividing prisms and truncated pyramids into pyramids and were challenged to find their volumes and realize that the sectioning methods were valid procedures.

I believe that space figures, volumes and surface areas are difficult topics for teachers to teach. This may be due to the fact that students have difficulty with the properties of triangle and other geometrical figures in the plane and when it comes to three dimensions they find them more difficult. When prospective teachers who take a course in geometry actually divide a prism or truncated pyramid into pyramids using the Geometer's Sketchpad and when they calculate volumes of the resulting pyramids and see that their sum

equals the volume of the original solids their visualization of three dimensions would be increased. They can share these exercises with their students and teach them visualization. The content of this paper can be useful in helping their students to overcome some of their difficulties as far as the three dimensional objects are concerned.

### Proofs in Geometry

Most proofs in geometry require us to show that line segments are congruent or angles are congruent. In order to do this we must identify two congruent triangles and show that the line segments or the angles are the corresponding parts of these congruent triangles. To identify the congruent triangles, occasionally we must draw auxiliary segments to form triangles. All this process has to be done visually as we set pencil on paper and make drawings. The visualization or imagination of the situation should precede any verbalization and drawing for the proof.

As in any learning activity, visualization has to be taught and geometry and three dimensional figures are rich areas for practicing visualization. The simple methods of dividing prisms into pyramids given in this paper provide students with problem-solving situations and practicing three-dimensional visualization.

### Polygons into Triangles

We can divide any polygon into triangles in many ways. For example:

- (1) Join any vertex of a polygon to other vertices. Figure 1(a)
- (2) Join any interior point of a polygon to the vertices. Figure 1(b)

In the first method the number of triangles will be  $n - 2$  with  $n > 3$  sides. In the second method the number of triangles will be  $n$ , the number of sides of the polygon.

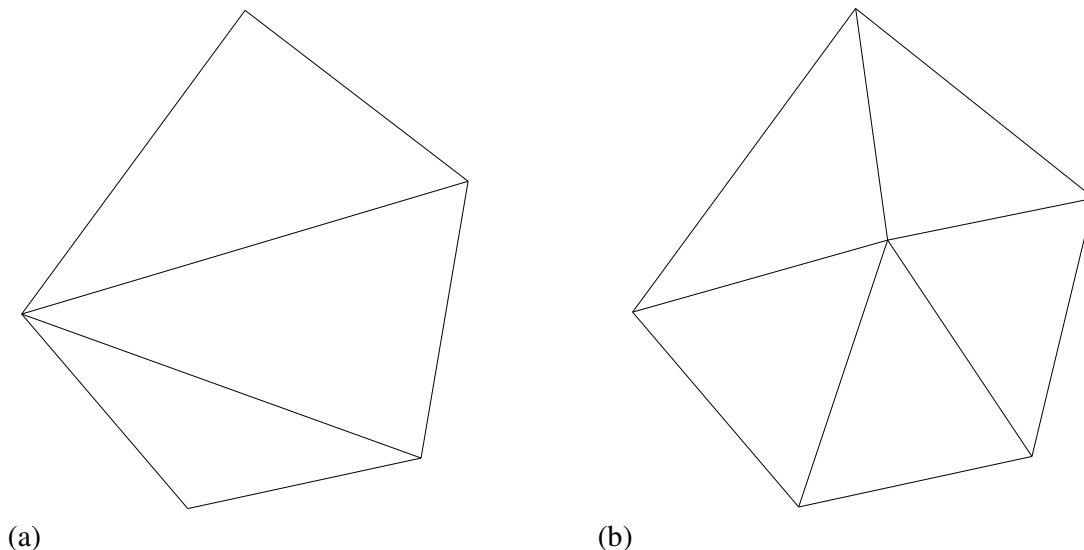
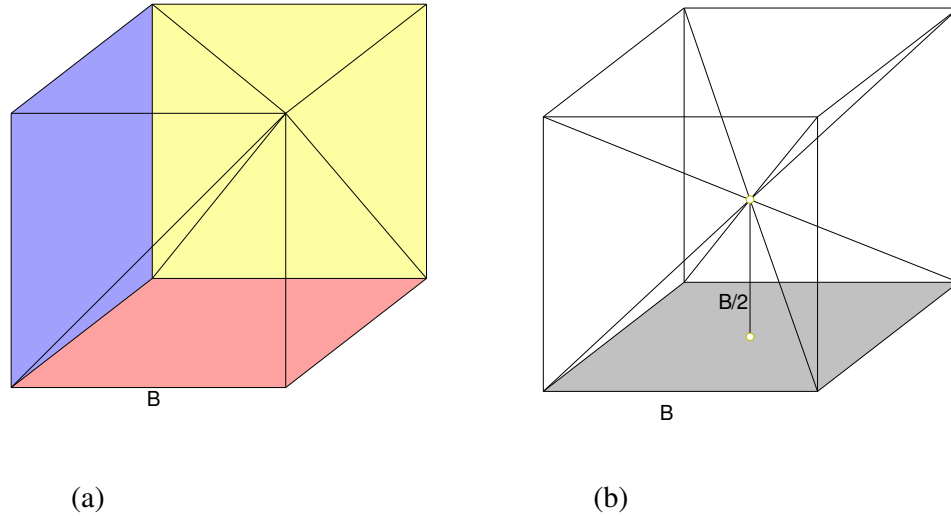


Figure 1

### Cubes into Pyramids

Let us generalize the above methods of dividing polygons to a cube of base length  $B$ . By the first method, the cube can be divided into three identical oblique pyramids of base length  $B$  by joining a vertex to all the other vertices. See Figure 2(a).



**Figure 2**

The volume of the cube with side  $B$  is  $B^3$ . Therefore, the volume of each square pyramid in Figure 2(a) is  $\frac{1}{3}B^3$ . In Figure 2(b), for simplicity, we have joined the center of the cube to each vertex and each pyramid formed on each face has volume  $\frac{1}{6}B^3$ .

### Square Prisms into Pyramids

The relationship between the volume of a square prism (cuboid, or rectangular parallelepiped) and square pyramid of the same height is demonstrated by filling a plastic model of the pyramid with water or sand and emptying it into the plastic model of the prism, [Posamentier/Smith/Stepelman 2006, p. 95]. The student will see that each time one-third of the prism is filled, and to fill the prism we must repeat this process three times. Therefore,

$$V_{\text{pyramid}} = \frac{1}{3}hB^2, \quad (1)$$

where,  $V_{\text{prism}} = hB^2$ , and the factor  $\frac{1}{3}$  for the volume of pyramid will be remembered. See Figure 3.

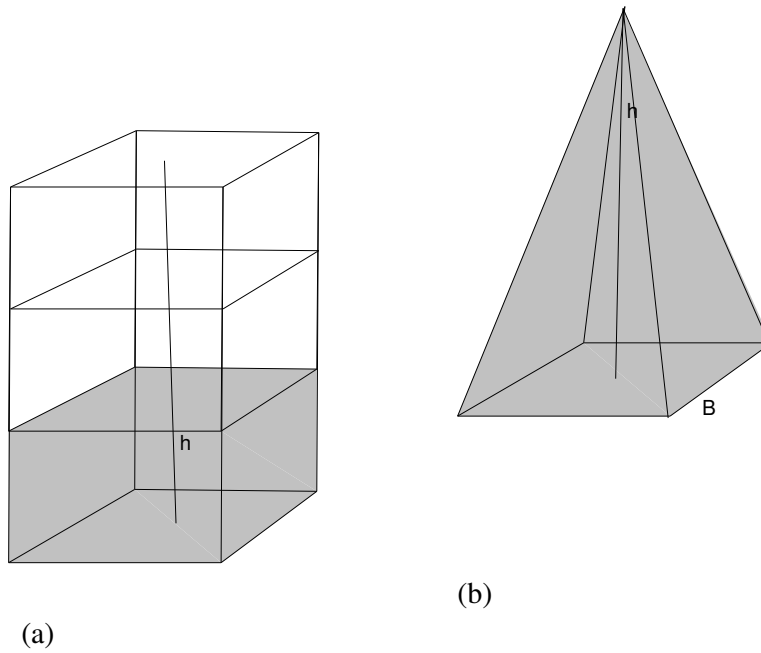


Figure 3

**Method 1**

A way to divide the square right prism is similar to the way that we divided the cube; by joining a vertex to all other vertices. See Figure 4.

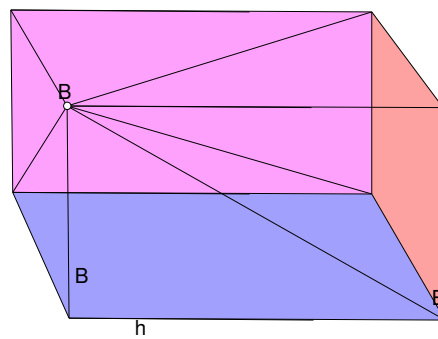


Figure 4

In this case, the sum of the volumes of the three pyramids is:

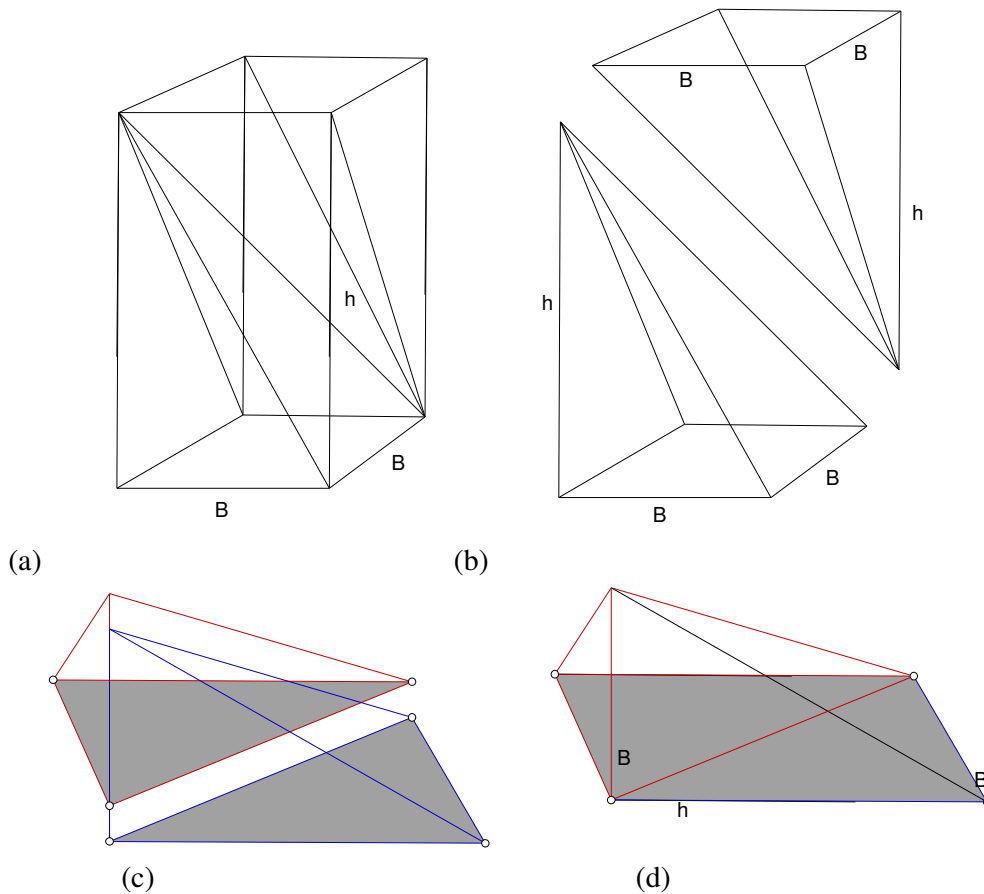
$$\begin{aligned} & \frac{1}{3} BhB + \frac{1}{3} BhB + \frac{1}{3} B^2h \\ &= \frac{1}{3} hB^2 + \frac{1}{3} hB^2 + \frac{1}{3} hB^2 \\ &= hB^2. \end{aligned}$$

**Remark**

We notice that as in the case of the cube, a pyramid is formed on the lower base, and the upper base is divided into two congruent triangles and each of these triangles is a face of one of the other two pyramids.

**Method 2**

Another way to divide the right square prism into oblique pyramids so that two of the pyramids have square bases is shown in Figure 5, below.



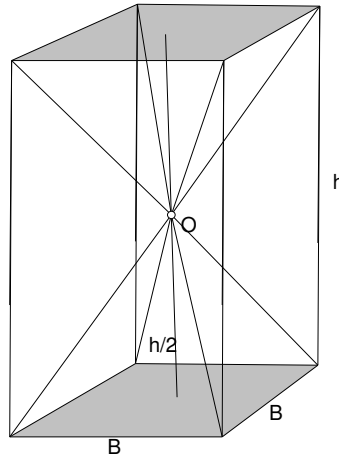
**Figure 5**

In Figure 5, the two triangular pyramids of vertical heights  $B$ , Figure 5(c), are formed inside the prism when we cut the two square pyramids, Figure 5(b) (with bases  $B^2$  and heights  $h$  and each of volume  $\frac{1}{3}hB^2$ ). We can put the two triangular pyramids together to form a rectangular pyramid, Figure 5(d) as shown and its volume is  $\frac{1}{3}B^2h$ .

To see the two triangular pyramids that are formed inside the prism, paper models of the pyramids were made and put together as in Photo 1. One student remarked that we could make a model of the square prism with play-doh, then cut the square pyramids and see the two triangular pyramids which will be left.

### Method 3

We can join an interior point of the square right prism to all its vertices and divide it into six pyramids, one on each face. See Figure 6. This is a generalization of the polygonal division into triangles, Figure 1(b).



**Figure 6**

In Figure 6, we have joined the midpoint of the altitude to all vertices to simplify the computation of the volumes of the six pyramids that are formed. One rectangular pyramid is formed on each face; and there are four of them, and each has volume  $\frac{1}{6}BBh = \frac{1}{6}hB^2$ . The other two pyramids have square bases and each has a volume of  $\frac{1}{6}hB^2$ . Their sum will be  $6(\frac{1}{6}hB^2) = hB^2$ , the volume of the square prism, with base side  $B$  and vertical height  $h$ .

### Pentagonal Right Prism

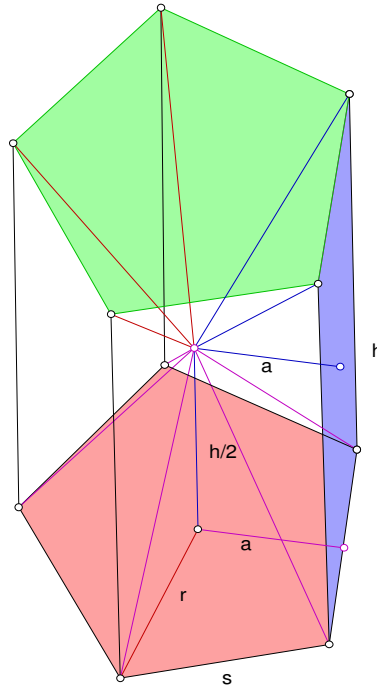
Next, let us consider a regular pentagonal prism of base side length  $s$  and height  $h$  whose volume is:

$$V = \frac{5}{2}has, \quad (2)$$

Where,  $h$  is the height of the prism,  $s$  is the base side length, and  $a$  is the apothem of the pentagonal base in Figure 7. In dividing the polygonal prisms into pyramids regularity is not necessary, and we are assuming that the base is a regular pentagon to simplify computation of the area of the base and hence the volume formula (2)

### Method 1

We can divide a pentagonal prism into polygonal pyramids by joining any point of the altitude, for example, the mid point of the altitude to the vertices of the lower base and the vertices of the upper base, as in Figure 7. This is again a generalization of dividing a polygon into triangles Figure 1(b). We describe this method first because trigonometry can be avoided in computations of the volumes.



**Figure 7**

Consequently, the above pentagonal prism is divided into two pentagonal pyramids with heights  $\frac{h}{2}$ , and five rectangular pyramids with base lengths  $s$  and  $h$  and heights the apothem  $a$  of the pentagon; a total of seven pyramids, altogether. In a polygonal prism with  $n$  base sides, this method results in  $n + 2$  pyramids

The two pentagonal pyramids and the five rectangular pyramids are shown in Figure 7, above. We will show that the volumes of these seven pyramids add up to the volume of the prism, formula (2).

Let us denote the area of each pentagonal base by  $A_1$ , the area of each lateral face by  $A_2$ , the volume of each of the pentagonal pyramids with height  $\frac{h}{2}$ , by  $V_1$ , and the volume of each of the rectangular pyramids with base area  $A_2$ , base sides  $s$ ,  $h$ , and height  $a$ , by  $V_2$ .

We see that  $A_1 = 5(\frac{1}{2}as)$  and  $A_2 = sh$  and

$$\begin{aligned}
 2V_1 + 5V_2 &= 2 \cdot \frac{h}{6} A_1 + 5 \cdot \frac{a}{3} A_2 \\
 &= 2 \cdot \frac{h}{6} (\frac{5}{2}as) + 5 \cdot \frac{a}{3} (sh) \\
 &= \frac{5}{6}has + \frac{5}{3}has \\
 &= \frac{5}{2}has \\
 &= V
 \end{aligned}$$

The above procedure can be applied to any polygonal prism.

## Method 2

We can also divide the pentagonal prism by joining a vertex of one of the bases to all other vertices, as seen in Figure 8, below. We note that this is a generalization of dividing a polygon into triangles by joining a vertex to all other vertices Figure 1(a).

There are four pyramids in figure 8. One is pentagonal with base  $ABCDE$ , the lower base of the prism and vertical height  $h$ , the height of the prism, and the other three are rectangular pyramids with bases,  $BCC'B'$ ,  $ABB'A'$  and  $AEE'A'$  all with base sides  $s$  and  $h$ . Each of the rectangular pyramids has a face that is one of the triangles that the upper pentagonal base of the prism is divided into, a generalization of the remark made in the case of the square prism. It remains to show that the volumes of these four pyramids add up to the volume of the pentagonal prism. First, we express  $a$  and  $s$  in terms of  $r$ . Where,  $r$  is the radius of the circumscribed circle of the pentagonal base. See Figure 7.

$$\begin{aligned} a &= r \cos 36^\circ \\ s &= 2r \sin 36^\circ \end{aligned} \quad (3)$$

It is necessary to use trigonometry since the rectangular pyramids have vertical heights that depend on the sine of the exterior angle of the triangular face. In the following, to simplify computations, besides formula (3) we also use the double-angle formula for sine,

$$\sin 72^\circ = 2 \sin 36^\circ \cos 36^\circ \quad (4)$$

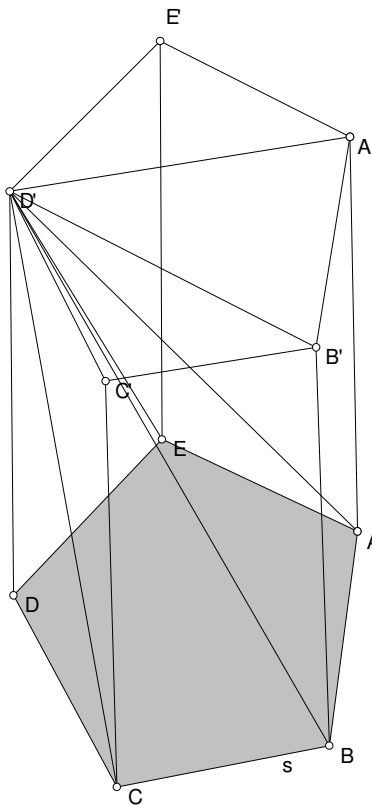


Figure 8



The pyramid with base  $ABCDE$  has volume  $V_1 = \frac{1}{3}h \cdot \frac{5}{2}as = \frac{5}{6}has$ . The pyramid with base  $BCC'B'$  has altitude  $s \cdot \sin 72^\circ = 2r \sin 36^\circ \sin 72^\circ$ , where we have substituted for  $s$  using formula (3). Therefore, its volume is

$$\begin{aligned} V_2 &= \frac{1}{3} \cdot 2r \sin 36^\circ \sin 72^\circ \cdot sh \\ &= \frac{1}{3} \cdot 2r \sin 36^\circ \sin 72^\circ \cdot 2r \sin 36^\circ h \\ &= \frac{4}{3}hr^2 \sin^2 36^\circ \sin 72^\circ \end{aligned}$$

We have substituted  $2r \sin 36^\circ$  for  $s$  in the second equation above and simplified the result. The pyramid with base  $ABB'A'$  has altitude

$$r + a = r + r \cos 36^\circ = r(1 + \cos 36^\circ)$$

We have used formula (3) again to substitute for  $a$ . Its volume is

$$\begin{aligned} V_3 &= \frac{1}{3}r(1 + \cos 36^\circ) \cdot sh \\ &= \frac{1}{3}r(1 + \cos 36^\circ) \cdot 2r \sin 36^\circ h \\ &= \frac{2}{3}hr^2(1 + \cos 36^\circ) \sin 36^\circ \end{aligned}$$

Finally, the volume of the pyramid with base  $AEE'A'$  is the same as the one with base  $BCC'B'$ , which is  $V_2$ . Applying the double-angle identity (4), the half-angle identity,

$$\sin^2 36^\circ = \frac{1 - \cos 72^\circ}{2},$$

and the identity for sine of supplementary angles,  $\sin 144^\circ = \sin(180^\circ - 144^\circ) = \sin 36^\circ$ , we proceed to simplify the sum of these volumes as follows:

$$V_1 + 2V_2 + V_3 = \frac{5}{6}hr^2 \sin 72^\circ + 2 \cdot \frac{4}{3}hr^2 \sin^2 36^\circ \sin 72^\circ + \frac{2}{3}hr^2(1 + \cos 36^\circ) \sin 36^\circ$$

$$\begin{aligned}
 &= \frac{5}{6}hr^2 \sin 72^\circ + \frac{8}{3}hr^2 \sin^2 36^\circ \sin 72^\circ + \frac{2}{3}hr^2 \sin 36^\circ + \frac{2}{3}hr^2 \cos 36^\circ \sin 36^\circ \\
 &= \frac{5}{6}hr^2 \sin 72^\circ + \frac{4}{3}hr^2(1 - \cos 72^\circ) \sin 72^\circ + \frac{2}{3}hr^2 \sin 36^\circ + \frac{1}{3}hr^2 \sin 72^\circ \\
 &= \frac{7}{6}hr^2 \sin 72^\circ + \frac{4}{3}hr^2 \sin 72^\circ - \frac{4}{3}hr^2 \cos 72^\circ \sin 72^\circ + \frac{2}{3}hr^2 \sin 36^\circ \\
 &= \frac{15}{6}hr^2 \sin 72^\circ - \frac{2}{3}hr^2 \sin 144^\circ + \frac{2}{3}hr^2 \sin 36^\circ \\
 &= \frac{5}{2}hr^2 \sin 72^\circ - \frac{2}{3}hr^2 \sin 36^\circ + \frac{2}{3}hr^2 \sin 36^\circ = \frac{5}{2}hr^2 \sin 72^\circ \\
 &= \frac{5}{2}hr^2 (2 \sin 36^\circ \cos 36^\circ) = \frac{5}{2}h(r \cos 36^\circ)(2r \sin 36^\circ) = \frac{5}{2}has.
 \end{aligned}$$

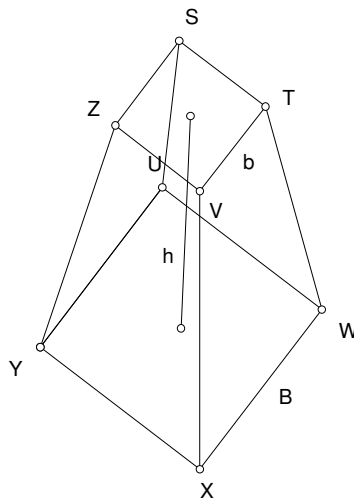
The last expression is the volume of the pentagonal prism, formula (2).

**Truncated Square Pyramid**

We now consider a truncated square pyramid (frustum) of lower base length  $B$ , upper base length  $b$  and vertical height  $h$ . See Figure 9, below. We note that a truncated square pyramid is not a prism, since the parallel bases are not congruent. However, our division procedures demonstrated above can be applied in this case because the volume formula for the truncated pyramid is the sum of volumes of pyramids. The formula is:

$$V = \frac{1}{3}h(B^2 + Bb + b^2). \tag{5}$$

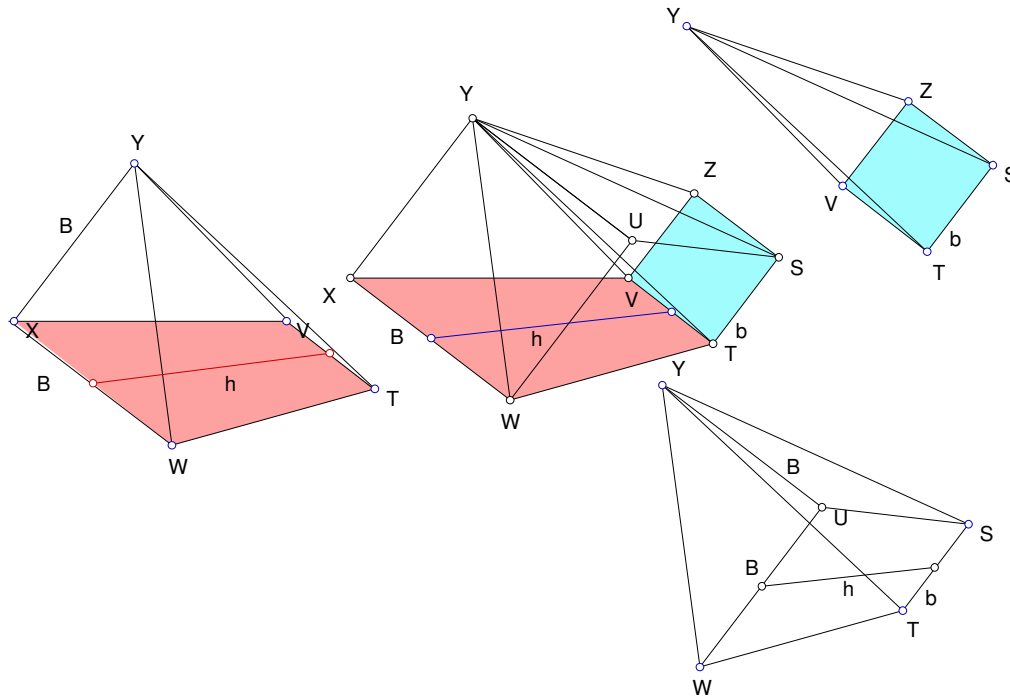
I have given a proof of formula (5) which appeared in [8], for other proofs and generalizations, see [1], [2], [5], and [6] in the References.



**Figure 9**

**Method 1**

As in the cases of a cube, square, and pentagonal prisms, we can divide the truncated square pyramid by joining a vertex of a base to all other vertices as shown in Figure 10, below. This again is a generalization of dividing a polygon into triangles by joining a vertex to all other vertices, Figure 1(a).



**Figure 10**

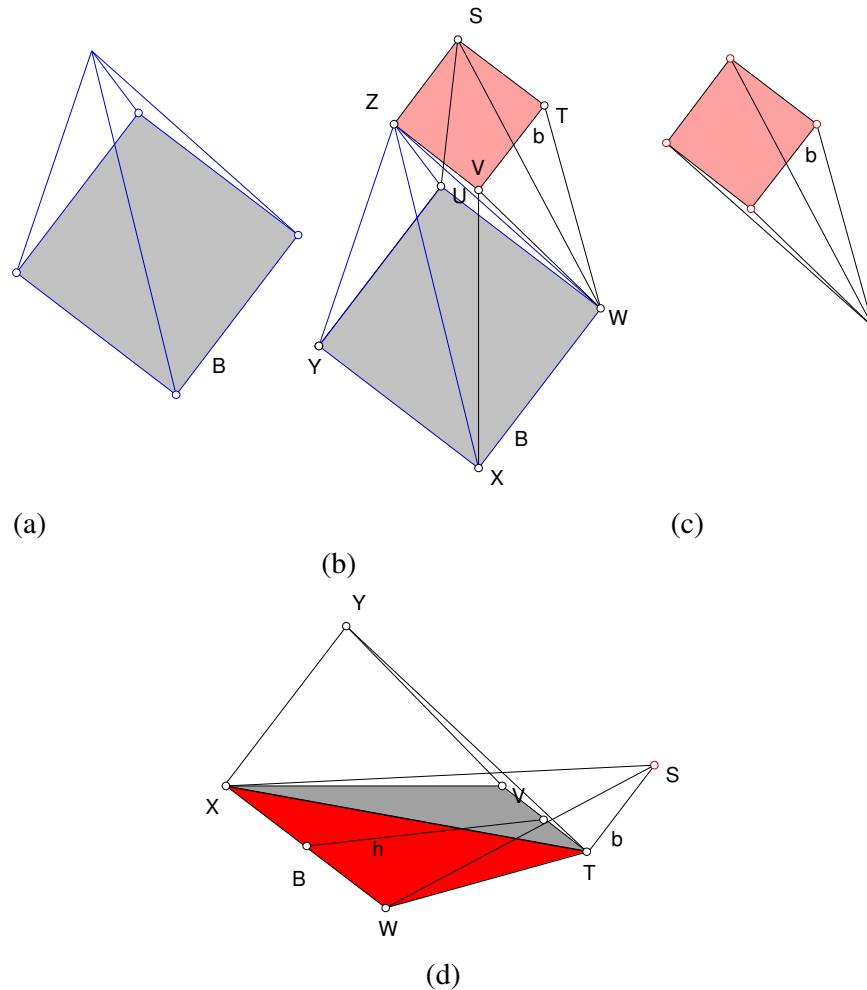
We find the volume of each oblique pyramid in figure 10 to see if they add up to the volume formula (5), above. The volume of the pyramid with the red trapezoidal base TWXV is  $\frac{1}{6}(B + b)hB$ . The volume of the one with the blue square base STVZ is  $\frac{1}{3}b^2h$ . Finally, the volume of the blank base pyramid with base the trapezoid STWU is also  $\frac{1}{6}(B + b)hB$ . Let us add these three volumes:

$$\begin{aligned} & \frac{1}{6}(B + b)hB + \frac{1}{3}b^2h + \frac{1}{6}(B + b)hB \\ &= \frac{1}{6}BhB + \frac{1}{6}bhB + \frac{1}{3}b^2h + \frac{1}{6}BhB + \frac{1}{6}bhB \\ &= \frac{1}{3}hB^2 + \frac{1}{3}hBb + \frac{1}{3}hb^2 \\ &= \frac{1}{3}h(B^2 + Bb + b^2). \end{aligned}$$

The last expression is V, the volume of the truncated square pyramid, formula (5).

**Method 2**

We can also divide the truncated square pyramid in the following way, Figure 11 below, because the volume formula (5) is the sum of volumes of two square pyramids and one rectangular pyramid.



**Figure 11**

As in the case of the square right prism, once we cut the two oblique square pyramids with base  $B^2$ , the lower base, and base  $b^2$ , the upper base of the truncated pyramid, Figures 11(a) and 11(c), two triangular pyramids Figure 11(d) with the base triangle TWX, and the base triangle TVX and altitudes  $b$  and  $B$ , respectively, are formed inside the truncated pyramid.

The volume of the first triangular pyramid is  $\frac{1}{3}b \cdot \frac{1}{2}Bh = \frac{1}{6}hBb$ , and the volume of the second is  $\frac{1}{3}B \cdot \frac{1}{2}bh = \frac{1}{6}hBb$ . The sum of these two volumes is,  $\frac{1}{6}hBb + \frac{1}{6}hBb = \frac{1}{3}hBb$ , which is the middle term in formula (5) for the volume of the truncated square pyramid.

**Method 3**

In Method 3 we apply a generalization of polygonal division Figure 1(b); the procedure outlined and demonstrated for prisms. We divide the truncated square pyramid into six pyramids by joining the midpoint of the altitude to all vertices.

In Figure 12 below, the midpoint of the altitude, O, is joined to the vertices of the upper base and the vertices of the lower base, and six pyramids are formed. We show that these six pyramids have volumes that add up to formula (5).

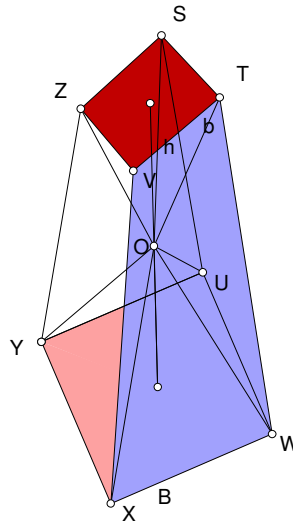


Figure 12

The lower and upper pyramids with bases  $B^2$  and  $b^2$ , respectively, and heights  $\frac{h}{2}$  have volumes

$$V_1 = \frac{1}{3} \cdot \frac{h}{2} B^2 \text{ and } V_2 = \frac{1}{3} \cdot \frac{h}{2} b^2. \text{ The four pyramids that are formed on the faces each have altitude } \frac{1}{4}(B + b).$$

So the volume of each is

$$\begin{aligned} V_3 &= \frac{1}{3} \cdot \frac{1}{4}(B + b) \cdot \frac{1}{2}(B + b)h = \frac{1}{24}h(B + b)^2 \\ &= \frac{1}{24}h(B^2 + 2Bb + b^2) \end{aligned}$$

Therefore, the sum of these four volumes is  $4V_3 = \frac{1}{6}h(B^2 + 2Bb + b^2)$ . Let us add all the six volumes.

$$\begin{aligned} V_1 + V_2 + 4V_3 &= \frac{1}{6}hB^2 + \frac{1}{6}hb^2 + \frac{1}{6}h(B^2 + 2Bb + b^2) \\ &= \frac{1}{3}hB^2 + \frac{1}{3}hBb + \frac{1}{3}hb^2 = \frac{1}{3}h(B^2 + Bb + b^2). \end{aligned}$$

The last expression is formula (5), the volume of the truncated square pyramid.

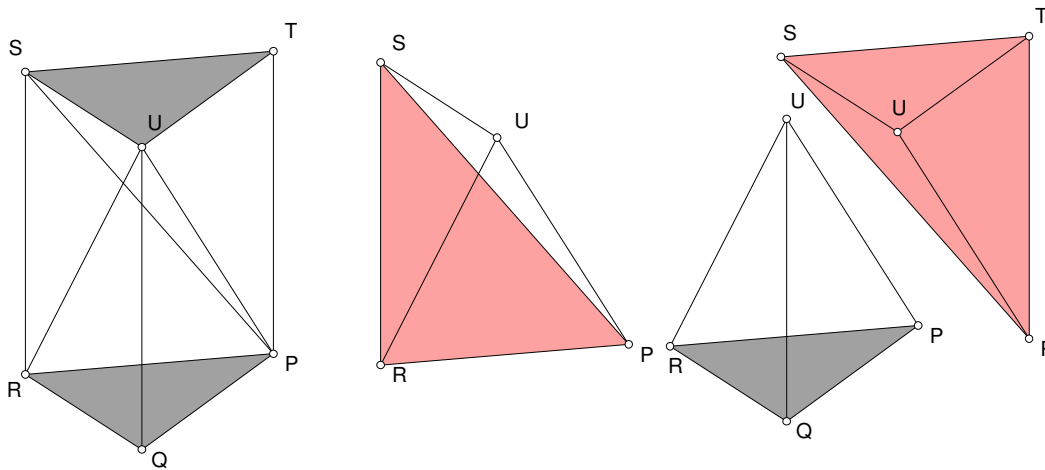
### Triangular Prisms

Finally, we consider a triangular prism. Its division into 3 and 5 pyramids is given below.

#### Method 1

The parallel bases of a triangular prism are triangles and we can only join the vertex  $P$  of the lower base to two vertices of the upper base,  $U$  and  $S$ . Then we join vertex  $U$  of the upper base to vertex  $R$  of the

lower base. Three triangular pyramids are formed which have equal volumes (Simmons 1987). See Figure 13, below.

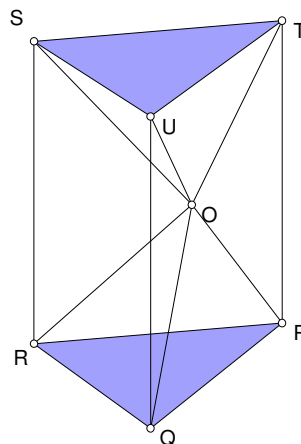


**Figure 13**

### Method 2

Method 2 is a generalization of dividing polygons into triangles by joining an interior point of the triangular prism to all the vertices. The triangular prism is divided into 5 pyramids; two triangular and three rectangular, one on each face. See Figure 14.

We can show that their volumes add up to the volume of the triangular prism the same way as in the square or pentagonal prisms.



**Figure 14**

### Conclusion

Students' evaluation of the above division methods was positive. They all said that their knowledge of three-dimensional geometry was increased and they learned how to find volumes of pyramids that make up a prism or a truncated pyramid. Moreover, they learned that we can divide polygonal prisms and truncated pyramids into polygonal pyramids in two general ways. First, we can join any interior point to the vertices of each face. In this way, a rectangular pyramid is formed on each face inside the prism. There are as many rectangular pyramids as there are faces. Two polygonal pyramids, one on each base, are also formed. Similarly, all truncated polygonal pyramids can be divided into pyramids, and the pyramids formed

on the faces are trapezoidal. In both cases, the number of pyramids will be  $n + 2$ , where  $n \geq 3$  is the number of sides of the base.

Second, if  $n > 3$ , we can join a vertex of a polygonal prism or a vertex of a truncated polygonal pyramid to all other vertices and this method will result in  $n - 1$  polygonal pyramids, where  $n$  is the number of sides of the base. One of the bases will be divided into  $n - 2$  triangles. Each of these triangles will be the face of a rectangular pyramid formed on a face in the case of a prism, or a trapezoidal pyramid formed on a face in the case of a truncated pyramid. There will be a polygonal pyramid with base the base of the prism or the base of the truncated polygonal pyramid, and the total is  $(n - 2) + 1 = n - 1$ . In the special case when the prism is triangular,  $n = 3$ , it is possible to divide it into 3 pyramids as in Figure 13.

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