Regular Total Domination In Line Graphs

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ABSTRACT

In this paper, we introduce the new concept in domination theory called regular total domination in line graphs. A dominating set D *of* $L(G)$ *is a regular total dominating set (RTDS) if the induced sub graph* $\langle D \rangle$ *has no isolated vertices and* $\deg(v)=1, \forall v \in D$. *The regular total domination number* $\gamma_{rt}(L(G))$ *is the minimum cardinality of a regular total dominating set. Also we study the graph theoretic properties of* $\gamma_{rt}(L(G))$ *and many bounds were obtained in terms of elements of* G *and its relationships with other domination parameters were found. Further, we show that the decision problem for regular total dominating set is an NP – complete even for bipartite graphs.*

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INTRODUCTION:

In this paper, we follow the notations of [1]. All the graphs considered here are simple and finite. As usual $p=|V|$ and $q=|E|$ denote the number of vertices and edges of a graph G respectively.

In general, we use $\langle X\rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ $(N[v])$ denote the open (closed) neighborhoods of a vertex v.

The notation $\alpha_0(G)$ $(\alpha_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. The notation $\beta_0(G)$ $(\beta_1(G))$ is the minimum number of vertices (edges) in a maximal independent set of a vertex (edge) of G. Let $\deg(v)$ is the degree of vertex v and as usual $\delta(G)$ $(\Delta(G))$ is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The degree of an edge $e=uv$ of G is defined by $\deg(e)=$ $\deg(u)+\deg(v)-2$ and $\delta'(G)\left(\Delta'(G)\right)$ is the minimum (maximum) degree among the edges of G. Further, we define private neighbor of u with respect to X, to be $pn[u,X]=\{v:N[v]\cap X=\{u\}\}$.

A line graph $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent (that is, are incident with a common vertex).

We begin by recalling some standard definitions from domination theory.

A set $S\subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V-S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is called the total dominating set, if for every vertex $v\in V$, there exists a vertex $u\in S$, $u\neq v$ such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G.

A set $D\subseteq V(L(G))$ is said to be dominating set of $L(G)$, if every vertex not in D is adjacent to a vertex in D. The domination number of $L(G)$ is denoted by $\gamma(L(G))$ is the minimum cardinality of a dominating set. The concept of domination in graphs with its many variations is now well studied in graph theory (see [2] and [3]).

In this paper, we introduce a new concept in domination theory. A dominating set D of $L(G)$ is a regular total dominating set (RTDS) if the induced subgraph $\langle D\rangle$ has no isolated vertices and $\deg(v)=1, \forall v\in D$. The regular total domination number $\gamma_{rt}(L(G))$ is the minimum cardinality of a regular total dominating set. Further, we study the graph theoretic properties of $\gamma_{rt}(L(G))$ and many bounds were obtained in terms of elements of G and its relationship with other domination parameters were found. Also we show that the decision problem for regular total dominating set (RTDS) is an NP – complete even for bipartite graphs.

RESULTS:

The following Theorem relates domination and regular total domination number of line graph of a tree in terms of edges of a tree.

Theorem 1: *For any connected* (p,q) *-tree* T *with at least two edges,* $\gamma_{rt}(L(T))+\gamma(L(T))\le q+1$.

Proof: First we start with the properties of RTDS of $L(T)$. Suppose, the regular total dominating set D of $L(T)$ is minimal, then for every vertex $x\in D$, the set $D-x$ does not dominate $L(T)$ or has an isolated vertex or both. In the first case, the vertex x has a D - private neighbor $p(x,D)\in N[x]-N[D-x]$.

In the second case, x is adjacent to an end vertex $u\in D$. If x itself is not an end vertex, then the set $D-\{u\}$ has no isolated vertex and thus u admits a D - private neighbor. Furthermore as u has a unique neighbor x in D. We now associate an injective way to each vertex x of $D^{'}$, where $D^{'}$ is the connected component of D without any D - private neighbor, another vertex u of $D^{'}$ which admits a D - private neighbor. Now since D is a minimal RTDS of $L(T)$, let D_1 be the minimal γ - set of $L(T)$ contained in D. The set D_1 is necessarily contains every vertex x of D which admits a D - private neighbor $p(x,D)$ and at least one vertex of each component of D which is isomorphic to K_2. Hence it follows that, $|D|\cup|D_1|\le q+1$. Therefore, $\gamma_{rt}(L(T))+\gamma(L(T))\le q+1$.

In the following Theorem, we give the upper bound for $\gamma_{rt}(L(T))$.

Theorem 2: *If every non end vertex of a* (p,q) *- tree* T *is adjacent to at least one end vertex, then* $\gamma_{rt}(L(T))\le p-m+1$, *where* m *is the number of end vertices in* T. *Equality holds if and only if* T *is isomorphic to a star.*

Proof: Let $A=\{v_1,v_2,...,v_m\}$ be the set of all end vertices in T with $|A|=m$. Suppose $V(T)-A=C$, then $N[v_i]=V(L(T))$, $1\le i\le n$, formed by the set of edges which are incident with the vertices of C. Let $D=\{v_1,v_2,...,v_k\}\subseteq V(L(T))$ be the minimum set of vertices which covers all the vertices in $L(T)$. Suppose $\deg(v_j)=1$, $\forall v_j\in D$, $1\le j\le k$, then D forms a minimal regular total dominating set of $L(T)$. Otherwise, if there exists at least one vertex $x\in D$ such that $\deg(x)=0$ in the subgraph $\langle D\rangle$. Then, we make $\deg(x)=1$ by adding a vertex $y\in N(x)$, such that $D\cup\{y\}$ forms a minimal γ_{rt}-set of $L(T)$. Clearly, it follows that $|D\cup\{y\}|\le p-|A|+1$. Therefore, $\gamma_{rt}(L(T))\le p-m+1$.

Suppose, T is isomorphic to a star $K_{1,n}$. Then in this case, $|D|=2$ and $|A|=p-1$. Therefore, it follows that $|D|=p-|A|+1$ and hence, $\gamma_{rt}(L(T))=p-m+1$.

Theorem 3: *For any* (p,q) *- tree* T, $\gamma_{rt}(L(T))+diam(T)\le p+\alpha_0(T)$.

Proof: Let $C=\{v_1,v_2,...,v_n\}\subseteq V(T)$, $\deg(v_i)\ge 2$, $\forall v_i\in C$, $1\le i\le n$, be the set of vertices which contains $C^{'}=\{v_1,v_2,...,v_k\}$, such that the set of vertices of $C^{'}$ covers all the edges in T. Now let $J=\{e_1,e_2,...,e_n\}\subseteq E(T)$ be the set of edges which are incident with the vertices of C. Clearly, there exists an edge set $J_1\subseteq J$ constituting the longest path between two distinct vertices $u,v\in V(T)$ such that $dist(u,v)=diam(T)$, the edge set J in T gives a vertex set in $L(T)$. Suppose $D=\{u_1,u_2,...,u_n\}\subseteq J$ be the minimum set of vertices which covers all the vertices in $L(T)$. Further, if each component of sub graph $\langle D\rangle$ is K_2, then D is a regular total dominating set of $L(T)$. Suppose there exists a vertex $x\in D$ such that $\deg(x)=0$ in $\langle D\rangle$. Then we make an attachment $y\in N(x)$ to make $\deg(x)=1$ such that $D\cup\{x,y\}$ forms a regular total dominating set of $L(T)$. Therefore, it follows that $|D\cup\{x,y\}|+diam(T)\le|V(T)|\cup|C^{'}|$ and hence

$\gamma_{rt}(L(T)) + diam(T) \le p + \alpha_0(T)$.

In the following Theorem, we give an upper bound to the regular total domination number of a line graph of a tree in terms of the elements of a tree.

Theorem 4: *For any* (p,q)*- tree* T, $\gamma_{rt}(L(T)) - 1 \le q - \alpha_1(T) + \left\lceil \frac{m}{3} \right\rceil$, *where* m *is the number of end vertices in* T.

Proof: Let $A = \{v_1, v_2, ..., v_n\} \subseteq V(T)$ be the set of all end vertices in T with $|A| = m$. Now if B be the set of edges which are incident with the edges of A. Then $B \cup F$, where $F \subseteq E(T) - B$, $dist(x, y) \ge 2$, $\forall x \in B$, $y \in F$, be the minimum set of edges which covers all the vertices in T, such that $|B \cup F| = \alpha_1(T)$. Now without loss of generality in $L(T)$, since $V(L(T)) = E(T)$, let $D = \{u_1, u_2, ..., u_k\} \subseteq B \cup F$ be the set of vertices such that $N[u_i] = V(L(T))$, $\forall u_i \in D$, $1 \le i \le k$. Further, if each component of the subgraph $\langle D \rangle$ is K_2. Then D forms a minimal regular total dominating set of $L(T)$. Otherwise, there exists at least one vertex $w \notin D$, such that $D \cup \{w\}$ forms a minimal γ_{rt}- set of $L(T)$. Clearly, it follows that $|D \cup \{w\}| - 1 \le |E(T)| - |B \cup F| \cup \left\lceil \frac{|A|}{3} \right\rceil$. Therefore, $\gamma_{rt}(L(T)) - 1 \le q - \alpha_1(T) + \left\lceil \frac{m}{3} \right\rceil$.

In the following Theorem we give lower bound to the regular total domination number of a line graph of a tree in terms of the elements of a tree.

Theorem 5: *For any* (p,q) *- tree* T, $\left\lceil \frac{p}{\Delta(T)+1} \right\rceil \le \gamma_{rt}(L(T)) + 1$.

Proof: Let D_1 be a minimal dominating set of $L(T)$ and $V_1 = V(L(T)) - D_1$. Suppose, $D_2 \subseteq V_1$, where $D_2 \in N(D_1)$, such that for every vertex $x \in D_1$, there exists a vertex $y \in D_1 \cup D_2$ or $y \in D_2$ and $\deg(v) = 1$ where $\forall v \in D_1 \cup D_2$ in the sub graph $\langle D_1 \cup D_2 \rangle$. Then $D_1 \cup D_2$ forms a minimal regular total dominating set of $L(T)$. Since for any tree T, there exists at least one vertex $u \in V(T)$ such that $\deg(u) = \Delta(T)$, it follows that $\left\lceil \frac{p}{\Delta(T)+1} \right\rceil \le |D_1 \cup D_2| + 1$. Hence, $\left\lceil \frac{p}{\Delta(T)+1} \right\rceil \le \gamma_{rt}(L(T)) + 1$.

Theorem 6: *For any* (p,q)*- tree* T, $\frac{q}{\Delta'(T)+1} \le \gamma_{rt}(L(T)) \le q - \delta'(T) + 1$. *Equality holds for* C_3 *and* $K_{1,n}$.

Proof: Let $e \in E(T)$ with $\deg(e) = \delta'(T)$. Now by definition of $L(T)$, $e = u \in V(L(T))$ and D be the minimal regular total dominating set of $L(T)$ such that $|D| = \gamma_{rt}(L(T))$. If $\delta'(T) \le 2$, then $|V(L(T))| \ge 2$ and $q - \gamma_{rt}(L(T)) \ge 2$. Clearly, $\gamma_{rt}(L(T)) \le q - 2 \le q - \delta'(T)$. Suppose, $\delta'(T) > 2$, then for any edge $e' \in N(e)$, $e' = w \in N(u)$ in $L(T)$ and $D \subseteq V(L(T)) - N(u) \cup \{w\}$, hence $\gamma_{rt}(L(T)) \le \left(q - (\delta'(T)+1) + 1\right)$. Therefore in any case, $\gamma_{rt}(L(T)) \le q - \delta'(T) + 1$.

Now, since every vertex in $V(L(T)) - D$ is adjacent to at least one vertex of D, hence each vertex in $V(L(T)) - D$ contributes at least one to the sum of the degrees of vertices of D and $|V(L(T)) - D| \le \sum_{v_i \in D} \deg v_i$. And the fact that for any tree T, there exists at least one edge $e \in E(T)$, $\deg(e) = \Delta'(T)$, we have $q-\gamma_{rt}\left(L(T)\right)=\left|V\left(L(T)\right)-D\right|\leq\sum_{v\in D}\deg v\leq\gamma_{rt}\left(L(T)\right)\cdot\Delta'(T)$. Therefore, $\frac{q}{\Delta^{'}(T)+1}\leq\gamma_{rt}\left(L(T)\right)$.

To extend our results from $\gamma_{rt}\left(L(T)\right)$ to $\gamma_{rt}\left(L(G)\right)$ in the following results, we define the following two types of operations.

Let $\Im$ be the family of $L(G)$ that can be obtained from either operation 1 or operation 2 listed below.

Operation 1: *The graph $L(G)$ is obtained from $L_1(G)$ by adding an edge xy to a vertex $x\in V\left(L_1(G)\right)$ belongs to some γ_{rt} - set of $L_1(G)$.*

Operation 2: *The graph $L(G)$ is obtained from $L_1(G)$ by adding an edge $e=uv$ to a vertex $u\in V(G)$ such that $e=x\in V\left(L_1(G)\right)$ belongs to some γ_{rt} - set of $L_1(G)$.*

By using above operations and family, we prove the following Lemmas.

Lemma 1: *Let $L_1(G)\in\Im$. If $L(G)$ is obtained from $L_1(G)$ by operation 1, then $L(G)\in\Im$.*

Proof: Let D be a γ_{rt}- set of $L_1(G)$ and $x\in D$. Suppose $L(G)$ is formed by attaching an edge xy to x. Then $D\cup\{x\}$ forms a minimal regular total dominating set of $L(G)$ and clearly, $L(G)\in\Im$.

Lemma 2: *Let $L_1(G)\in\Im$. If $L(G)$ is obtained from $L_1(G)$ by operation 2, then $L(G)\in\Im$.*

Proof: Let D be a γ_{rt}- set of $L_1(G)$ and $x\in D$. Suppose $L(G)$ is formed such that by attaching an edge uv to $x=e\in E(G)$ at u in G. Then the corresponding component incident with u forms K_3 in $L(G)$ and $D\cup\{y\}$, $y\in N(x)$, forms a minimal regular total dominating set of $L(G)$. Clearly, $L(G)\in\Im$.

By using above two Lemmas we prove the following Theorem.

Theorem 7: *Let D_{rt} be the minimum regular total dominating set of a line graph $L(G)$ belonging to the family $\Im$, then D_{rt} is the unique minimum regular total dominating set of $L(G)$.*

Proof: We proceed by induction on the number n of operations on $L(G)$ required to construct the line graph $L(G)$. If $n\left(L(G)\right)=0$ then obviously, the result holds. Assume now that $L(G)$ is a line graph with $n\left(L(G)\right)=k$ for some positive integer k and for each $L_1(G)\in\Im$ with $n\left(L_1(G)\right)<k$, the result is true. Then $L(G)$ is obtained from $L_1(G)$ belonging to $\Im$ by operation 1 or operation 2. We now consider the following two cases depending on whether $L(G)$ is obtained from $L_1(G)$ by operation 1 or operation 2.

Case 1: $L(G)$ is obtained from $L_1(G)$ by operation 1. Then x is an end vertex in $L(G)$ and $y\in N(x)$. Thus x and y belongs to every regular total dominating set of $L(G)$. By the induction hypothesis, there is exactly one minimum regular total dominating set $D^{'}$ in $L_1(G)$ and each component of $\left\langle D^{'}\right\rangle$ is a K_2. Since y is in some $\gamma_{rt}\left(L_1(G)\right)$- set, we conclude that there is exactly one minimum regular total dominating set in $L(G)$, which is $D=D^{'}\cup\{x,y\}$ and each component of $\langle D\rangle$ is K_2.

Case 2: $L(G)$ is obtained from $L_1(G)$ by operation 2, then there exists a non end vertex $w\in L(G)$ and $y\in N(w)$ such that w and y belonging to every regular total dominating set of $L(G)$. Since x is in no $\gamma_{rt}\left(L_1(G)\right)$- set, there is exactly one minimum regular total dominating set in $L_1(G)$, denoted by $D^{'}$, and each component of $\left\langle D^{'}\right\rangle$ is K_2, we conclude that there is exactly one minimal regular total dominating set in

$L(G)$, which is $D = D^{'} \cup \{y, w\}$ and each component of $\langle D \rangle$ is K_2.

The following Theorem relates domination number of a graph and regular total domination number of its line graph.

Theorem 8: *For any connected* (p,q) *- graph* G *with* $p \geq 3$ *vertices,* $\gamma_{rt}(L(G)) + \gamma(G) \leq p+1$.

Proof: Suppose $L(G) \in \Im$ and $C = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the set of all vertices with $\deg(v_i) \geq 2$, $\forall v_i \in C$, $1 \leq i \leq n$. Then there exists a minimal set $C^{'} \subseteq C$ which covers all the vertices in G. Clearly, $C^{'}$ forms a minimal γ- set of G. Now without loss of generality in $L(G)$, $F = \{u_1, u_2, ..., u_n\} \subseteq V(L(G))$ be the set of vertices in $L(G)$ corresponding to the edges which are incident with the vertices of $C^{'}$ in G. Further, there exists a set $D = \{u_1, u_2, ..., u_m\} \subseteq F$ such that $N[u_j] = V(L(G))$, $\forall u_j \in D$, $1 \leq j \leq m$ and $\deg(u_j) = 1$ in the subgraph $\langle D \rangle$ of $L(G)$. Clearly, D forms a minimal regular total dominating set of $L(G)$. Hence it follows that $|D| \cup |C^{'}| \leq p+1$. Therefore, $\gamma_{rt}(L(G)) + \gamma(G) \leq p+1$.

If $L(G) \notin \Im$, then $L_1(G)$ is obtained from $L(G)$ either by operation 1 or operation 2. Further, by Lemma 1 and 2, $L_1(G) \in \Im$. Since $L_1(G)$ is obtained from $L(G)$, $\gamma_{rt}(L_1(G)) \geq |D| - 1 = \gamma_{rt}(L(G)) -1$. However γ_{rt}- set of $L_1(G)$ can be extended to γ_{rt}- set of $L(G)$ by the operation 1 or operation 2. Hence $D - \{v\}$ be the γ_{rt}- set of $L(G)$ and therefore, $|D - \{v\}| \cup |C'| \leq p+1$. Clearly, $\gamma_{rt}(L(G)) + \gamma(G) \leq p+1$.

The following Theorem relates total domination number of a graph and regular total domination number of its line graph.

Theorem 9: *For any connected* (p,q) *- graph* G, $\gamma_{rt}(L(G)) + \gamma_t(G) \leq p + \delta(G) + 1$.

Proof: Suppose $L(G) \in \Im$ and let S be a γ_t- set of G. By the minimality, for any vertex $v \in S$, the subgraph $\langle S - v \rangle$ contains an isolated vertex. Let $S_1 = \{v : v \in S\}$ and A be the set of all end vertices in $\langle S_1 \rangle$, $B = S_1 - A$. Further, let C be the minimum set of vertices of $S - S_1$ such that each vertex of A is adjacent to some vertex of C. Clearly, $|C| \leq |A|$. Now let $S^{'} = S - \{S_1 \cup C\}$ and every $u_i v_i \in \langle S^{'} \rangle$, $1 \leq i \leq k$, such that $|S^{'}| = \gamma_t(\langle S^{'} \rangle)$. Hence $S^{'}$ forms a smaller total dominating set of G. Now by definition of $L(G)$, let $F = \{v_1, v_2, ..., v_n\} \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident to the vertices of $S^{'}$ in G. Let $D = \{v_1, v_2, ..., v_k\} \subseteq F$, such that for every $x_j \in D$, there exists exactly one neighbor $y_j \in D$, $1 \leq j \leq k$, be the set of vertices which covers all the vertices in $L(G)$. Clearly, D forms a minimal γ_{rt}- set of $L(G)$. Since for any graph G, there exists at least one vertex $v \in V(G)$, such that $\deg(v) = \delta(G)$, it follows that $|D| \cup |S^{'}| \leq p + \delta(G) + 1$. Therefore, $\gamma_{rt}(L(G)) + \gamma_t(G) \leq p + \delta(G) + 1$.

If $L(G) \notin \Im$, then $L_1(G)$ is obtained from $L(G)$ either by operation 1 or operation 2. Further, by Lemma 1 and 2, $L_1(G) \in \Im$. Since $L_1(G)$ is obtained from $L(G)$, $\gamma_{rt}(L_1(G)) \geq |D| - 1 = \gamma_{rt}(L(G)) -1$. However γ_{rt}- set of $L_1(G)$ can be extended to γ_{rt}- set of $L(G)$ by the operation 1 or operation 2. Hence $D - \{v\}$ be the γ_{rt}- set of $L(G)$ and therefore, $|D - \{v\}| \cup |S'| \leq p + \delta(G) + 1$. Clearly, $\gamma_{rt}(L(G)) + \gamma_t(G) \leq p + \delta(G) + 1$.

In the following Theorem we give an upper bound to the regular total domination number of a line graph.

Theorem 10: *For any connected (p,q) - graph G, $\gamma_{rt}(L(G)) \le 2(p-\beta_0(G))$. Equality holds if $G \cong K_{1,n}$.*

Proof: Suppose $L(G) \in \Im$ and let $K = \{v_1, v_2, ..., v_k\} \subseteq V(G)$ be the maximum set of vertices with $N(u) \cap N(v) = \{x\}$, $\forall u, v \in K$ and $x \in V(G) - K$. Clearly, K forms a maximal independent set of vertices. Now let $L = \{e_1, e_2, \;\; ..., e_k\} \subseteq E(G)$ be the set of edges which are incident with the vertices of K. Clearly, L forms a vertex set in $L(G)$ by the definition of line graph. Let $D = \{v_1, v_2, ..., v_n\} \subseteq L$ in $L(G)$ be the set of vertices which covers all the vertices in $L(G)$. Suppose $\deg(v_i) = 1$, $\forall v_i \in D$, $1 \le i \le n$ in the sub graph $\langle D \rangle$, then D itself is a regular total dominating set of $L(G)$. Otherwise, make $\deg(v_i) = 1$ by adding vertices $\{u_i\} \in V(L(G)) - D$ and $\{u_i\} \in N(v_i)$. Clearly, $D \cup \{u_i\}$ forms a minimal regular total dominating set of $L(G)$. Therefore, it follows that $|D \cup \{u_i\}| \le 2(|V(G)| - |K|)$ and hence $\gamma_{rt}(L(G)) \le 2(p - \beta_0(G))$.

If $L(G) \notin \Im$, then $L_1(G)$ is obtained from $L(G)$ either by operation 1 or operation 2. Further, by Lemma 1 and 2, $L_1(G) \in \Im$. Since $L_1(G)$ is obtained from $L(G)$, $\gamma_{rt}(L_1(G)) \ge |D| - 1 = \gamma_{rt}(L(G)) \;\; -1$. However γ_{rt} - set of $L_1(G)$ can be extended to γ_{rt} - set of $L(G)$ by the operation 1 or operation 2. Hence $D - \{v\}$ be the γ_{rt} - set of $L(G)$ and therefore, $|D - \{v\}| \le 2(|V(G)| - |K|)$. Clearly, $\gamma_{rt}(L(G)) \le \; 2(p - \beta_0(G))$.

Suppose, $G \cong K_{1,n}$. Then in this case, $|D| = 2$ and $|K| = p - 1$. Clearly, it follows that $\gamma_{rt}(L(G)) = 2(p - \beta_0(G))$.

To show that the decision problem for arbitrary graphs is NP – complete, we need to use a well known NP – completeness result, called Exact Three Cover (X3C), which is defined as follows.

EXACT COVER BY 3 – SETS (X3C):

Instance: A set X with $|X| = 3q$ and a collection $\Re$ of 3 – elements subsets of X.

Question: Is $\Re$ contains an exact three cover for X, that is a sub collection $\Re' \subseteq \Re$ such that every element of X occurs in exactly one member of $\Re'$? Note that if $\Re'$ exists, then its cardinality is precisely q.

REGULAR TOTAL DOMINATING SET (RTDS):

Instance: A line graph $L(G)$ and a positive integer $k \le |V(L(G))|$.

Question: Is there a regular total dominating set of cardinality at most k for the line graphs belonging to the family $\Im$?

Theorem 11: *RTDS is NP – complete, even for the line graphs of bipartite graphs.*

Proof: It is obvious that RTDS is a member in NP, since we can in polynomial time guess at a partition of $V(L(G))$ and in polynomial time verify that each set is a RTDS of $L(G)$. To show that RTDS of $L(G)$ is an NP – complete problem when restricted to bipartite graphs, we will establish a polynomial transformation from X3C. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $\Re = \{R_1, R_2, ..., R_m\}$ be an arbitrary instance of X3C, we will construct a line graph $L(G)$ of bipartite graph and a positive integer k such that this instance of X3C will have an exact three cover if and only if $L(G)$ has a RTDS of cardinality at most k.

We now describe the construction of $L(G)$ for bipartite graph G. Corresponding to each variable

$x_i \in X$ associate a path $x_i y_i z_i$. Corresponding to each R_j associate a K_2 with vertices c_j and d_j. The construction of $L(G)$ for a bipartite graph G is completed by joining x_i and c_j if and only if $x_i \in R_j$. Finally, the set $k = m + 4q$.

Suppose, $\Re$ has an exact 3 – cover, say $\Re'$. Then $\bigcup_{i=1}^{3q}\{z_i\} \cup \bigcup_{j=1}^{m}\{d_j\} \cup \{c_j / c_j \in \Re'\}$ is a RTDS of cardinality $m+4q$. This construction can clearly be accomplished in polynomial time.

Conversely, suppose D is RTDS of cardinality at most $m+4q$ in $L(G)$. Then the vertices in the set A, defined by $\bigcup_{i=1}^{3q}\{z_i\} \cup \bigcup_{j=1}^{m}\{d_j\}$ have to be in D. Hence $|D| - |A| \le (m+4q) - (m+3q) = q$. Let $B = \{ i \in \{1,2,...,3q\} / x_i \in D$ or $y_i \in D\}$ and $B_1 = \{ j \in \{1,2,...,m\} / c_j \in D\}$. Then, since D is a γ- set of $L(G)$, $\left(\bigcup_{i \in B}\{x_i, y_i\} \cup \bigcup_{j \in B_1} N[c_j]\right) \cap \{x_1, x_2, ..., x_{3q}\} \supseteq \{x_1, x_2, ..., x_{3q}\}$. We conclude that $|B| \cup |B_1| \ge 3q$. Also $|B| \cup |B_1| \le |D| - |A| \le q$. Hence $3(|B| \cup |B_1|) \le |B| \cup 3|B_1|$, so that $B = \phi$, $x_i, y_i \notin D$ for $i = 1,2,...,3q$. Since x_i, $i = 1,2,...,3q$, is dominated by D, we must have $|B_1| = q$ and that $\Re' = \{c_i / j \in B_1\}$ is an exact cover for X.

Theorem 12: *RTDS is NP – complete.*
Proof: The proof of this Theorem is similar to the proof of Theorem11.

Finally, we give the inequality chain which is straight forward,

Theorem 13: *For any connected graph* G, $\gamma(L(G)) \le \gamma_t(L(G)) \le \gamma_{rt}(L(G))$.

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