OPTIMISATION OF A FLAT PUNCH WITH POLYNOMIAL CROSS-SECTION

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Abstract

In macro and micro contacts, a punch having a flat end contacting an elastic half-space generates a highly non-uniform contact pressure which tends to infinity on the contour of contact area. The increase of carrying capacity of such contacts requires a more uniform distribution of contact pressure. This can be obtained by adequately profiling the front surface of the punch end. This paper points out a procedure to derive that end surface of a punch with a simply-connected cross-section which yields an uniform contact pressure distribution.

Keywords: rigid punch, uniform contact pressure, polygonal cross-section

1. Introduction

In contact mechanics a contact between two bodies can be modeled by a suitably profiled rigid punch which cotacts normally an elastic half-space. There are many non-Hertzian surface contacts which require a flat end punch model. In such a contact, the pressure reaches a finite value in the centre of contact area but increases towards infinity as the observed point gets closer to the bounding contour, [1 - 4]. This high pressure drastically reduces the carrying capacity of the contact.

In order to increase the applicable normal load of a contact, a more uniform theoretical contact pressure is required. A solution to this problem may consist in replacing the flat end of the punch by a slightly crowned surface. According to Feijoo and Fancello, [5], and Hasslinger, [6], an optimum situation implies the generation of a constant contact pressure over the whole contact area. Consequently, the optimization of a contact requires the derivation of that end surface of the punch which yields an uniform distribution of contact pressure.

This paper presents a simple method to derive this optimum end surface for a rigid punch having a simply-connected cross-section. This surface is expressed as a simple definite integral or as a sum of such integrals having as integrands certain characteristic lengths of the cross-section. This method is applied for a few polygonal cross-sections.

2. Preliminary elements

Let a rigid punch bounded by a front surface $z(x, y)$ be in contact with an elastic half-space of elastic parameters E (Young modulus) and ν (Poisson ratio). These parameters define a contact stiffness by following equation:

$$\eta = \frac{1-\nu^2}{E} \tag{1}$$

The co-ordinates of all points placed on the contact surface fulfill the following integral condition of deformation:

$$\frac{\eta}{\pi}\iint_A \frac{p(x', y')dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} = \delta - z(x, y), \tag{2}$$

where A is contact area, p is contact pressure, δ stands for the normal approach between the contacting bodies, in its turn equal to normal displacement of the origin of the half-space along z axis and x', y' represent auxiliary co-ordinates parallel, to x and y axes, respectively.

It is then imposed that for a certain nominal load Q_n over the contact area, which coincides with the punch cross section, acts a constant nominal pressure p_n, $p(x, y) = p(x', y') = p_n$. As a result, the following equation for that end surface of the punch which generates a constant pressure over the contact area can be written as:

$$z(x, y) = \delta - \frac{\eta}{\pi} p_n \iint_A \frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} \tag{3}$$

in which δ is given by:

$$\delta = \frac{\eta}{\pi} p_n \iint_A \frac{dx'dy'}{\sqrt{(x')^2+(y')^2}} \tag{4}$$

In the above equations nominal pressure is equal to the mean pressure, in its turn defined as the ratio of nominal load Q_n to contact area, A: $p_n = p_m = Q_n / A$.

Equation (3) can be solved analytically or numerically in order to find the required end surface of the punch under nominal load.

The actual shape of the surface is hidden in a normal plot. Therefore, it seems more convenient to turn the punch upside down in order to make its end surface visible. Mathematically, this reversal is simply obtained by subtracting the ordinate z from an arbitrary constant length. This length is arbitrarily chosen equal with contour ordinate z_C named maximum recession of front surface.

In the present study, simply-connected cross-sections bounded by several curve arcs which are described by different equations will be taken into account and a general method of integration will be envisaged. Then, this will be applied to several usual polygonal cross-sections.

3. Simply-connected cross-section: multiple equation contour

A single connected cross-section of a punch bounded by several curve arcs described by various equations, as shown in Fig. 1, is considered. A particular integration procedure can be envisaged for the integral entering the Eq. (3). The current point M in which the ordinate z of the end surface is calculated and one of bounding arcs, denoted by i, form a sector. Adjacent points of the curve arc are L_i and L_{i+1}. This sector is divided by *MN*, the normal *MN*, to this arc passing through point M, in two smaller sectors, denoted by j, $j = 1, 2$.

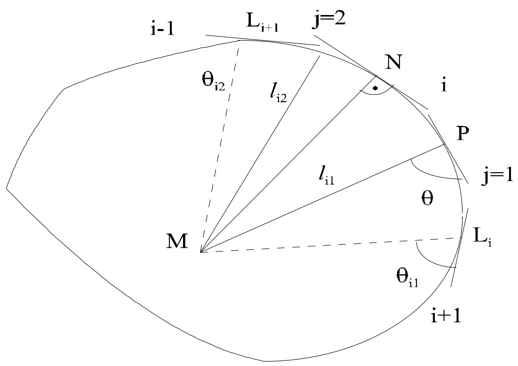

Figure 1. Auxiliary construction

This way the contact area is divided into $2n$ sub domains. As a result, an integral over area A becomes a sum of $2n$ integrals over these sub domains. Each area integral can be transformed into a definite integral if a straight line segment is traced from M to a current point P of the arc bounding the considered sub domain. Let ℓ_{ij} and θ be the length of this segment and the angle it forms with the tangent to the contour in P, respectively. Any sub domain integral which corresponds to Eq. (3) takes the form:

$$I_{ij} = \int_{\theta_{ij}}^{\pi/2} \ell_{ij}\, d\theta\,,\ j = 1,2 \qquad (5)$$

θ_{ij} being the angle between the segment ML_i or ML_{i+1} and tangent in each vertex L_i, respectively, L_{i+1} of the considered sector. Consequently, the contribution of each sector i to the integral entering Eq. (3) can be written as follows:

$$I_i = I_{i1} + I_{i2} = I_{ij} = \int_{\theta_{i1}}^{\theta_{i2}} \ell_{i1}\, d\theta + \int_{\theta_{i2}}^{\pi/2} \ell_{i2}\, d\theta \qquad (6)$$

and the whole integral becomes:

$$I = \sum_{i=1}^{n} I_i \qquad (7)$$

4. Polygonal cross-section

4.1. General method

A polygon with n sides is considered, as shown in Fig. 2, from which one side i, located between sides $i+1$ and $i-1$, is isolated. This polygon is refereed to a co-ordinate system xy, with x axis directed along one edge and y axis passing through a vertex so that the polygon is placed in the first quadrant. The vertexes bounding side i are $V_{i1} = (x_{i1}, y_{i1})$ and $V_{i2} = (x_{i2}, y_{i2})$. The length of this side is ℓ_i. Each vertex has two symbolizations, corresponding to the two adjacent sides, namely $V_{i1} \equiv V_{(i-1)2}$ and $V_{i2} \equiv V_{(i+1)1}$. A point $M(x, y)$ is considered on the contact area A in which the optimum ordinate z of the end surface is calculated.

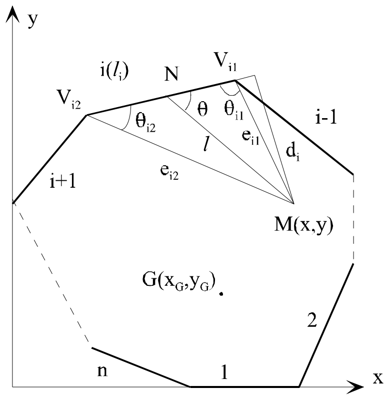

Figure 2. Polynomial cross-section with n sides

Let $d_i(x,y)$ be the distance from $M(x,y)$ to side *i* and $e_{i1}(x,y)$ and $e_{i2}(x,y)$ the lengths of segments $\overline{MV_{i1}}$ and $\overline{MV_{i2}}$. If ℓ is the distance from point $M(x,y)$ to point N on side *i*, then the integrals involved in Eqs. 3 and 4 become sums of simple integrals expressed in terms of the angle *θ*, formed between segment $\overline{MN}$ $\overline{MN}$ and side *i* of the polygon. As a result, Eqs. (3) and (4) are expressed by simple integrals as follows:

$$z(x,y)=\delta-\frac{\eta}{\pi}p_n\sum_{i=1}^{n}d_i(x,y)\left[\int_{\theta_{i1}(x,y)}^{\pi/2}\frac{d\theta}{\sin\theta}+\int_{\theta_{i2}(x,y)}^{\pi/2}\frac{d\theta}{\sin\theta}\right] \quad (8)$$

$$\delta=\frac{\eta}{\pi}p_n\sum_{i=1}^{n}d_i(x_G,y_G)\left[\int_{\theta_{i1}(x_G,y_G)}^{\pi/2}\frac{d\theta}{\sin\theta}+\int_{\theta_{i2}(x_G,y_G)}^{\pi/2}\frac{d\theta}{\sin\theta}\right] \quad (9)$$

where x_G, y_G are the co-ordinates of the center of gravity G of area A. Integration limits $\theta_{i1}(x,y)$ and $\theta_{i2}(x,y)$ are obtained by applying the cosine theorem in the triangle formed by the points, M, I_1, I_2, resulting in:

$$\left.\begin{aligned}\theta_{i1}(x,y)&=\left|a\cos\left(\frac{e_{i1}^2(x,y)+\ell_i^2-e_{i2}^2(x,y)}{2e_{i1}(x,y)\,\ell_i}\right)\right|\\ \theta_{i2}(x,y)&=\left|a\cos\left(\frac{e_{i2}^2(x,y)+\ell_i^2-e_{i1}^2(x,y)}{2e_{i2}(x,y)\,\ell_i}\right)\right|\end{aligned}\right\} \quad (10)$$

Lengths involved in Eqs. (10) can be expressed in terms of vertices co-ordinates as follows:

$$\left.\begin{aligned}\ell_i&=\sqrt{\left(x_{i2}-x_{i1}\right)^2+\left(y_{i2}-y_{i1}\right)^2}\\ e_{ij}(x,y)&=\sqrt{\left(x-x_{ij}\right)^2+\left(y-y_{ij}\right)^2}\end{aligned}\right\} \quad (11)$$

To avoid possible singularities that may appear in expressions of slope, m_i, and distance, d_i, when one or two adjacent sides of the polygon are parallel to the y axis it is preferable to express these parameters by aid of special functions "if", as follows:

$$m_i=if\left[x_{i1}=x_{i2},\infty,\frac{y_{i2}-y_{i1}}{x_{i2}-x_{i1}}\right] \quad (12)$$

$$d_i(x,y)=if\left[x_{i1}=x_{i2},\left|x-x_{i1}\right|,\frac{\left|m_ix-y-n_i\right|}{\sqrt{1+m_i^2}}\right] \quad (13)$$

where n_i is the cut determined in z axis by a line containing side i:

$$n_i=y_{i1}-m_ix_{i1} \quad (14)$$

In the equation above, the x, y co-ordinates must correspond to a point of the contact area A. To ensure this, it is necessary to use a function $g(x, y)$ unitary within area A, contour included, and zero outside of A.

The equation of side *i* of the polygon can be written as follows:

$$y = m_i x + n_i = m_i (x - x_{i1}) + y_{i1} \tag{15}$$

A partial function $g_i(x, y)$ can be written for each side *i* as follows:

$$g_i(x, y) = if\left[\left(y - y_{i1} - m_i\left(x - x_{i1}\right)\right)\cdot\left(y_C - y_{i1} - m_i\left(x_C - x_{i1}\right)\right) \geq 0, 1, 0\right] \tag{16}$$

where index C stands for points belonging to the cross section contour. This function is equal to the unit when point $M(x, y)$ belongs to that part of the half-plane containing area A and is zero when $M(x, y)$ is placed in the opposite side of the half-plane. Therefore, function $g(x, y)$ is defined as product of partial functions $g_i(x, y)$:

$$g(x, y) = \prod_i g_i(x, y) \tag{17}$$

Finally, optimum punch front surface takes the form:

$$Z(x, y) = z(x, y) g(x, y) \tag{18}$$

Simple integrals from Eqs. (8), (9) and (18) can be calculated numerically by various methods, one of which is that provided by the Mathcad software. Because this program considers $\infty = 10^{307}$ and the slope in Eq. (16) is multiplied by the various abscissa differences which may lead to values higher than 10^{307} is convenient to write this equation in the following form:

$$g_i(x, y) = if\left[\frac{y - y_{i1} - \dfrac{m_i(x - x_{i1})}{\rho_k}}{\left|y - y_{i1} - \dfrac{m_i(x - x_{i1})}{\rho_k}\right|}\cdot\frac{y_C - y_{i1} - \dfrac{m_i(x_C - x_{i1})}{\rho_k}}{\left|y_C - y_{i1} - \dfrac{m_i(x_C - x_{i1})}{\rho_k}\right|} \geq 0, 1, 0\right] \tag{19}$$

where $\rho_k = x_k / N$, N represents the chosen number of discretizations and k the current range on the abscissa.

This general procedure is applied below to find the optimum end surface for punches which possess either arbitrary triangular or hexagonal cross-sections. In order to obtain generally valid results, it is more convenient to use dimensionless co-ordinates defined by following equations:

$$\left.\begin{aligned} \bar{\delta} &= \delta \frac{\pi}{a\eta p_n} \\ \bar{z}_C &= z_C \frac{\pi}{a\eta p_n} \\ \bar{z} &= z \frac{\pi}{a\eta p_n} \end{aligned}\right\} \tag{20}$$

4.2. Triangular cross-section

An arbitrary triangle reported to a co-ordinate system with the origin in a vertex and the *x* axis directed along one of its sides, adjacent to the origin is considered. For example, dimensionless co-ordinates of vertexes $(0,0)$, $(5,0)$ and $(3,4)$ are considered. Their substitution in the above general equations lead to tree-dimensional optimum end surface of the punch shown in Fig. 3. Dimensionless recessions of the triangle vertices are 4.304, 4.697 and 4.697. Therefore, for this triangle, $\bar{z}_C = 4.697$ and occurs in the vertices of co-ordinates $(5,0)$ and $(3,4)$. By summation of the maximum ordinate 6.249 with $\bar{z}_C$ the dimensionless normal approach between the contacting bodies $\bar{\delta} = \bar{z}_C + 6.249 = 10.964$ is obtained.

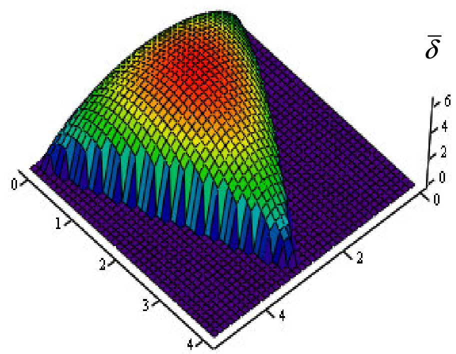

Figure 3. Optimum end surface of a arbitrary triangular punch

4.3. Regular hexagon

A regular hexagon with unitary dimensionless sides is considered. Dimensionless coordinates of the vertices, covered clockwise, are: $(0,0.5\sqrt{3})$, $(0.5,0)$, $(1.5,0)$, $(2,0.5\sqrt{3})$, $(1.5,\sqrt{3})$ and $(0.5,\sqrt{3})$. Optimum end surface of the punch, obtained numerically, is shown in Fig. 4.

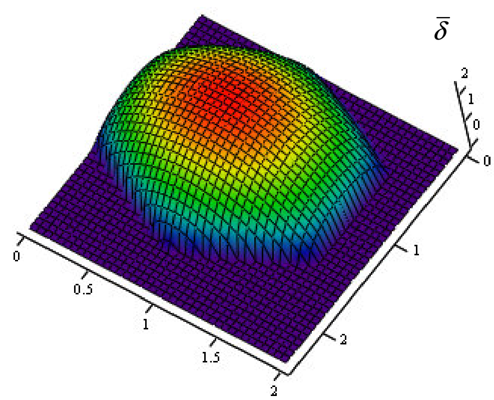

Figure 4. Optimum end surface of a regular hexagon punch

All vertices ordinates are equal and have the value 3.232, yielding a dimensionless normal approach $\bar{\delta} = 5.708$.

The same method can be applied for other cross sections such as regular and irregular convex polygons or multiple-connected contours.

Conclusions

1. The idea of replacing a flat front surface of a rigid punch that contacts an elastic half-space by a slightly curved one was exploited above in order to generate an uniform pressure distribution over the whole contact area.

2. The equation of this optimum end surface is expressed in an integral form. Consequently, the ordinate of this surface was derived numerically by using a Surface Mathcad file.

3. General applicability of the results is provided for by means of using dimensionless co-ordinates. This solution shows that a general optimum end surface of contacting end of a punch cannot exist

independently of load. Such a surface yields from an imposed value of the contact pressure, that is, from an unique normal load.

4. The crowning of an optimum end surface is very small and, consequently, such a surface can be machined on digitally-controlled machine tools only.

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