A NEW INTEGRAL OPERATOR AND SOME OF ITS PROPERTIES

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Abstract. In this paper we will prove, using Becker criterion, the univalence of the integral operator $T(f_i, g)(z)$, considered for analitic functions f_i and g in the open unit disk $\mathcal{U}$. Starting from this operator, we will present some properties for it.

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1. Introduction and Preliminaries

Let the unit disk $\mathcal{U} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the class of univalent and regular functions.

Definition 1. [5] *We say that a function $f \in \mathcal{A}$ is in the class of starlike functions of order β, denoted by $\mathcal{S}^*_\beta$, if it satisfies:*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathcal{U}, \tag{1}$$

for $\beta \in [0, 1)$.

Definition 2. [4] *We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{R}_\beta$, if it satisfies:*

$$Re\left(f'(z)\right) > \beta, \quad z \in \mathcal{U}, \tag{2}$$

for $\beta \in [0, 1)$.

Definition 3. [7] *We say that a function* $f \in \mathcal{A}$ *is in the class* $\mathcal{N}(\beta)$*, if it satisfies:*

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right)<\beta, \quad z\in\mathcal{U}, \tag{3}$$

for $\beta \in \mathbb{R}$, $\beta > 1$.

Definition 4. [2] *We say that a function* $f \in \mathcal{A}$ *is in the class* $\mathcal{B}(\mu,\beta)$, $\mu \geq 0$, $\beta \in [0,1)$*, if it satisfies:*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\beta, \quad z\in\mathcal{U}. \tag{4}$$

The following sufficient condition for univalency of an analytic function in the unit disk was given by Becker in [1]:

Theorem 1. *Let* $f \in \mathcal{A}$*. If for all* $z \in \mathcal{U}$ *we have:*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right|\leq 1 \tag{5}$$

then the function f *is univalent in* $\mathcal{U}$.

Also, an important result that we will use in our paper is General Schwarz Lemma. We remind it here:

Lemma 2. [3] *Let the regular function* f *in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} \mid |z| < R\}$*, with* $|f(z)| < M$*,* M *fixed. If* f *has in* $z = 0$ *one zero with multiply* $\geq m$*, then*

$$|f(z)|\leq\frac{M}{R^m}|z|^m\ , \quad z\in\mathcal{U}_R. \tag{6}$$

It is obviously that for $R = m = 1$ the relation (6) becomes:

$$|f(z)|\leq M|z|\ , \quad z\in\mathcal{U}. \tag{7}$$

The goal of our paper is to introduce an integral operator, to prove the univalence for it and present some properties obtained from here.

2. Main results

For a function $f : Z \to Z$, we denote by $f^{(k)}$ its derivative of order k.

Theorem 3. *Let $f_i, g \in \mathcal{A}$, $i = \overline{1,n}$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha \geq 1$, $M \geq 1$ and N_i be positive real numbers.*

If we have:

i) $\left|\frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right| \leq N_i, \ z \in \mathcal{U}$;

ii) $|g'(z)| \leq M, \ z \in \mathcal{U}$;

iii) $|\alpha| \leq \frac{3\sqrt{3}}{2\left(M+\sum\limits_{i=1}^{n} N_i\right)}$,

then the function:

$$T(f_i, g)(z) = \int_0^z \left(e^{g(t)} \prod_{i=1}^{n} f_i^{(i)}(t)\right)^{\alpha} dt \tag{8}$$

is in the class $\mathcal{S}$.

Proof. We have:

$$T'(f_i, g)(z) = \left(e^{g(z)} \prod_{i=1}^{n} f_i^{(i)}(z)\right)^{\alpha}$$

and

$$\begin{aligned} T''(f_i, g)(z) &= \alpha \left(e^{g(z)} \prod_{i=1}^{n} f_i^{(i)}(z)\right)^{\alpha-1} \left(e^{g(z)} \sum_{k=1}^{n} \left(f_k^{(k+1)}(z) \prod_{\substack{i=1 \\ i\neq k}}^{n} f_i^{(i)}(z)\right)\right. \\ &\quad \left. + e^{g(z)} g'(z) \prod_{i=1}^{n} f_i^{(i)}(z)\right) \\ &= \alpha \left(e^{g(z)} \prod_{i=1}^{n} f_i^{(i)}(z)\right)^{\alpha-1} e^{g(z)} \prod_{i=1}^{n} f_i^{(i)}(z) \left(g'(z) + \sum_{k=1}^{n} \frac{f_k^{(k+1)}(z)}{f_k^{(k)}(z)}\right) \\ &= \alpha \left(e^{g(z)} \prod_{i=1}^{n} f_i^{(i)}(z)\right)^{\alpha} \left(g'(z) + \sum_{k=1}^{n} \frac{f_k^{(k+1)}(z)}{f_k^{(k)}(z)}\right). \end{aligned}$$

So we obtain:

$$\frac{T''(f_i, g)(z)}{T'(f_i, g)(z)} = \alpha \left(g'(z) + \sum_{i=1}^{n} \frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right), \tag{9}$$

hence:

$$\left(1 - |z|^2\right) \left|\frac{zT''(f_i, g)(z)}{T'(f_i, g)(z)}\right| = \left(1 - |z|^2\right) |z| \, |\alpha| \left|g'(z) + \sum_{i=1}^{n} \frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right|.$$

We can consider the function:

$$h(|z|)=\left(1-|z|^2\right)|z|,$$

as a real function defined on $[0;1)$, because $|z|<1$, and it is easy to obtain the maximum value of $h(|z|)$ on $[0,1)$ as being $\frac{2}{3\sqrt{3}}$.

So we have:

$$\left(1-|z|^2\right)\left|\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)}\right|\le\frac{2}{3\sqrt{3}}|\alpha|\left|g'(z)+\sum_{i=1}^{n}\frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right|$$

Using properties i) and ii) for the function f and g, we have:

$$\left(1-|z|^2\right)\left|\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)}\right|\le\frac{2}{3\sqrt{3}}|\alpha|\left|M+\sum_{i=1}^{n}N_i\right|$$

and by condition iii) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)}\right|\le 1.$$

Hence, by Becker univalence criterion, we proved that the operator $T(f_i,g)(z)$ is in the class $\mathcal{S}$.

Considering $N_i=M=1$, for every $i=\overline{1,n}$, in Theorem 3, we obtain the following corollary:

Corollary 4. *Let* $f_i,g\in A$, $i=\overline{1,n}$, *with* $\mathrm{Re}\alpha\ge 1$. *If:*

i) $\left|\frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right|\le 1,\ z\in\mathcal{U}$;

ii) $|g'(z)|\le 1,\ z\in\mathcal{U}$;

iii) $|\alpha|\le\frac{3\sqrt{3}}{2(n+1)}$,

then the function $T(f_i,g)(z)$ *defined in (8) is in the class* $\mathcal{S}$.

Considering $f_i(z)=\frac{z^i}{i!}$, for $i=\overline{2,n}$, and $N_1=1$ we have the following result obtained by Ularu and Breaz in [6]:

Theorem 5. [6] *Let* α *be a complex number with* $\mathrm{Re}\alpha\le 1$, M *be a positive real number* $(M\ge 1)$ *and the functions* $f,g\in A$. *If:*

i) $\left|\frac{f''(z)}{f'(z)}\right|\le 1, z\in U$;

ii) $|g'(z)|\le M, z\in U$;

iii) $|\alpha|\le\frac{3\sqrt{3}}{2(M+1)}$,

then the function

$$I_1(f,g)(z)=\int_0^z\left(f'(z)e^{g(t)}\right)^{\alpha}dt$$

is in the class $\mathcal{S}$.

Example 1. The function:

$$T_1(z) = \int_0^z \left(\prod_{p=1}^{n} \left(1 + \sum_{k=1}^{p} \frac{1}{k!} t^k \right) e^t \right)^{\alpha} dt,$$

with $p \in \mathbb{N}^*$, is an univalent function if $|\alpha| \leq \frac{3\sqrt{3}}{2\left(M + \sum_{i=1}^{n} N_i\right)}$.

Indeed, if we consider $f_p(z) = z + \sum_{k=1}^{p} \frac{1}{(k+1)!} z^{k+1}$, $p \in \mathbb{N}^*$, and $g(z) = z$, it will be easy to verify the conditions from Theorem 3.

Theorem 6. *Let $f_i, g \in \mathcal{A}$, $i = \overline{1,n}$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}\alpha \geq 1$, $M \geq 1$ and N_i be positive real numbers.*

If we have:

i) $\left| \frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)} \right| \leq N_i,\ z \in \mathcal{U}$;

ii) $g \in \mathcal{B}(\mu, \beta),\ \mu \geq 0,\ \beta \in [0,1)$;

then the operator given by (8) belongs to $\mathcal{N}(\rho)$, where:

$$\rho = |\alpha| \left(\sum_{i=1}^{n} N_i + (2-\beta) M^{\mu} \right) + 1$$

Proof. According to Deffinition 3 We will prove that the function $T(f_i, g)(z)$ satisfies:

$$\mathrm{Re}\left(\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)} + 1 \right) < \rho, \quad z \in \mathcal{U} \tag{10}$$

From (9), we have:

$$\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)} = \alpha z \left(\sum_{i=1}^{n} \frac{f_i^{(i+1)}}{f_i^{(i)}} + g'(z) \right)$$

We will obtain successively:

$$\begin{aligned} \mathrm{Re}\left(\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)} + 1 \right) &= \mathrm{Re}\left[\alpha z \left(\sum_{i=1}^{n} \frac{f_i^{(i+1)}}{f_i^{(i)}} + g'(z) \right) + 1 \right] \\ &< |\alpha|\,|z| \left(\left| \sum_{i=1}^{n} \frac{f_i^{(i+1)}}{f_i^{(i)}} \right| + \left| g'(z) \right| \right) + 1 \end{aligned}$$

Because $|z| < 1$ and i) we have:

$$\begin{aligned} \mathrm{Re}\left(\frac{zT''(f_i,g)(z)}{T'(f_i,g)(z)} + 1 \right) &< |\alpha| \left(\sum_{i=1}^{n} N_i + \left| g'(z) \right| \right) + 1 \\ &= |\alpha| \left(\sum_{i=1}^{n} N_i + \left| g'(z) \left(\frac{z}{g(z)} \right)^{\mu} \right| \left| \frac{g(z)}{z} \right|^{\mu} \right) + 1 \\ &\overset{Schwarz}{<} |\alpha| \left(\sum_{i=1}^{n} N_i + \left(\left| g'(z) \left(\frac{z}{g(z)} \right)^{\mu} - 1 \right| + 1 \right) M^{\mu} \right) + 1 \\ &\overset{g \in B(\mu,\beta)}{<} |\alpha| \left(\sum_{i=1}^{n} N_i + (2-\beta) M^{\mu} \right) + 1 \\ &= \rho. \end{aligned}$$

So we proved the inequality (10) and we can conclude that $T(f_i, g)(z)$ is in the class $\mathcal{N}(\rho)$.

Taking $\mu = 0$ in Theorem 6, we have:

Corollary 7. *Let $f_i, g \in \mathcal{A}$, $i = \overline{1,n}$, $\alpha \in \mathbb{C}$ with* $\mathrm{Re}\alpha \geq 1$, $M \geq 1$ *and N_i be positive real numbers.*

If we have:

i) $\left|\frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right| \leq N_i,\ z \in \mathcal{U};$

ii) $g \in \mathcal{R}_\beta,\ \beta \in [0,1);;$

then the operator given by (8) belongs to $\mathcal{N}(\rho)$, where:

$$\rho = |\alpha|\left(2 - \beta + \sum_{i=1}^{n} N_i\right) + 1.$$

If we consider $\mu = 1$ in Theorem 6, we obtain:

Corollary 8. *Let $f_i, g \in \mathcal{A}$, $i = \overline{1,n}$, $\alpha \in \mathbb{C}$ with* $\mathrm{Re}\alpha \geq 1$, $M \geq 1$ *and N_i be positive real numbers.*

If we have:

i) $\left|\frac{f_i^{(i+1)}(z)}{f_i^{(i)}(z)}\right| \leq N_i,\ z \in \mathcal{U};$

ii) $g \in \mathcal{S}^*_\beta,\ \beta \in [0,1);;$

then the operator given by (8) belongs to $\mathcal{N}(\rho)$, where:

$$\rho = |\alpha|\left((2 - \beta)M + \sum_{i=1}^{n} N_i\right) + 1.$$

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