On the Approximation of the Step function by Raised-Cosine and Laplace Cumulative Distribution Functions

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Abstract
In this note the Hausdorff approximation of the Heaviside step function by Raised-Cosine and Laplace Cumulative Distribution Functions arising from lifetime analysis, financial mathematics, signal theory and communications systems is considered and precise upper and lower bounds for the Hausdorff distance are obtained. Numerical examples, that illustrate our results are given, too.

Key words: Heaviside step function, Raised-Cosine Cumulative Distribution Function, Laplace Cumulative Distribution Function, Alpha–Skew–Laplace cumulative distribution function, Hausdorff distance, upper and lower bounds

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1. Introduction

The Cosine Distribution is sometimes used as a simple, and more computationally tractable, approximation to the Normal Distribution. The Raised-Cosine Distribution Function (RC.pdf) and Raised-Cosine Cumulative Distribution Function (RC.cdf) are functions commonly used to avoid inter symbol interference in communications systems [1], [2], [3]. The Laplace distribution function (L.pdf) and Laplace Cumulative Distribution function (L.cdf) is used for modeling in signal processing, various biological processes, finance, and economics. Examples of events that may be modeled by (L.pdf) include: credit risk and exotic options in financial engineering, uncurance claims and structural changes in switching–regime model and Kalman filter [4].

Definition 1. The RC.cdf \( f(t; a, b) \) is defined for \( a - b < t < a + b, a \in \mathbb{R}, b > 0 \) by:

\[
f(t; a, b) = \frac{1}{2} \left( 1 + \frac{t-a}{b} + \frac{1}{\pi} \sin \left( \frac{(t-a)\pi}{b} \right) \right).
\]  

(1)

Special case. For \( a=0 \) we obtain the special RC.cdf:

\[
f(t; 0, b) = \frac{1}{2} \left( 1 + \frac{t}{b} + \frac{1}{\pi} \sin \left( \frac{t\pi}{b} \right) \right),
\]  

for which \( f(0; 0, b) = \frac{1}{2} \).

Definition 2. The L.cdf \( f_1(t; a, b) \) is defined for \( b>0 \) by:

\[
f_1(t; a, b) = \begin{cases} 
\frac{1}{2} e^{\frac{t-a}{b}}, & \text{if } t < a, \\
1 - \frac{1}{2} e^{\frac{a-t}{b}}, & \text{if } t \geq a,
\end{cases}
\]  

(3)

where \( a > 0 \) is a location parameter.

Definition 3. The Heaviside step function \( h_0(t) \) is defined by:

\[
h_0(t) = \begin{cases} 
0, & \text{if } t < 0, \\
[0, 1], & \text{if } t = 0 \\
1, & \text{if } t > 0
\end{cases}
\]  

(4)

2. Approximation of the Heaviside function by RC.cdf.

We study the Hausdorff approximation [5] of the step function \( h_0(t) \) by RC.cdf of the form (2) and find an expression for the error of the best approximation.

The Hausdorff distance \( d=d(b) \) between the function \( h_0(t) \) and the function \( f(t; 0, b) \) satisfies the relation

\[
f(d; 0, b) = \frac{1}{2} \left( 1 + \frac{d}{b} + \frac{1}{\pi} \sin \left( \frac{d\pi}{b} \right) \right) = 1 - d,
\]  

or

\[
\frac{1}{2\pi} \sin \frac{d\pi}{b} + \frac{d}{2b} - \frac{1}{2} + d = 0.
\]  

(5)

(6)
The following proposition gives upper and lower bounds for \( d(b) \).

**Proposition 1.** The Hausdorff distance \( d = d(b) \) between the step function \( h_0 \) and the RC.cdf(2) can be expressed in terms of the parameter for any real \( 0 < b < 2.524 \) as follows:

\[
\frac{1}{2.5 \left( 1 + \frac{1}{b} \right)} < d < \frac{\ln \left( \frac{2.5 \left( 1 + \frac{1}{b} \right)}{2.5 \left( 1 + \frac{1}{b} \right)} \right)}{2.5 \left( 1 + \frac{1}{b} \right)}
\]

(7)

**Proof.** Let us examine the function (see, (6))

\[
F(d) = \frac{1}{2\pi} \sin \frac{d\pi}{b} + \frac{d}{2b} - \frac{1}{2} + d.
\]

From \( F'(d) > 0 \) we conclude that the function \( F \) is strictly monotonically increasing. Consider function

\[
G(d) = -\frac{1}{2} + \left( 1 + \frac{1}{b} \right) d.
\]

Figure 1: The cumulative function (2) with \( b = 0.5 \); Hausdorff distance \( d = 0.178351 \).

Figure 2: The cumulative function (2) with \( b = 0.1 \); Hausdorff distance \( d = 0.0575502 \).
From Taylor expansion
\[
d - \frac{1}{2} + \frac{d}{2b} + \frac{1}{2\pi} \sin \frac{d\pi}{b} = -\frac{1}{2} + \left(1 + \frac{1}{b}\right) d + O(d^3)
\]
We obtain \( G(d) - F(d) = O(d^3) \).

Hence \( G(d) \) approximates \( F(d) \) with \( d \rightarrow 0 \) as \( O(d^3) \)(see, Fig. 3).

In addition \( G'(d) > 0 \). Further, for \( 0 < b < 2.524 \) we have

\[
G\left(\frac{1}{2.5 \left(1 + \frac{1}{b}\right)}\right) = -\frac{1}{2} + \frac{1}{2.5} < 0,
\]

\[
G\left(\frac{\ln \left(2.5 \left(1 + \frac{1}{b}\right)\right)}{2.5 \left(1 + \frac{1}{b}\right)}\right) = -\frac{1}{2} + \frac{1}{2.5} \ln \left(2.5 \left(1 + \frac{1}{b}\right)\right) > 0.
\]

This completes the proof of the proposition.

Some computational examples using relation (6) are presented in Table 1 for various \( b \).

3. Approximation of the shifted Heaviside function by L.cdf.
We study the Hausdorff approximation [5] of the shifted step function by L.cdf of the form (3) and find an expression for the error of the best approximation.
Print["Calculation of the value of the Hausdorff distance $d$ between the Raised-Cosine cumulative distribution function and the Heaviside step function:"]

b = Input[" b "]; (*0.05 *)
Print[" The parameter $b = ", b];
Print["The following nonlinear equation is used to determination of the Hausdorff distance $d$: "];
n[d_] := d - 1/2 + d/(2*b) + (1/(2*Pi))*Sin[(d*Pi)/b];
Print[n[d], " =0"];
Print["The unique positive root of the equation is the searched value of $d$: "];
FindRoot[n[d], {d, 0.001}]
Print[TableForm[S]];

Calculation of the value of the Hausdorff distance $d$ between the Raised-Cosine cumulative distribution function and the Heaviside step function

The parameter $b = 0.05$

The following nonlinear equation is used to determination of the Hausdorff distance $d$:

$$\frac{1}{2} + 11.1d + \frac{\sin(62.8319d)}{2\pi} = 0$$

The unique positive root of the equation is the searched value of $d$:

$\{d \rightarrow 0.0326076\}$

Figure 4: Simple module implemented in programming environment *Mathematica* for calculation of the value of the Hausdorff distance $d$ between the Heaviside step function and the sigmoidal RC.cdf function.

Manipulate[Dynamic@Show[Plot[f[t], {t, -0.05, 0.05}, LabelStyle -> Directive[Green, Bold],
    PlotLabel -> (1/2)*(1 + t/b + (1/Pi)*Sin[(t*Pi)/b])],
    PlotRange -> Automatic, {0, 0.99}], AxesOrigin -> {0, 0}], {{b, 0}, 0, 30,
    Appearance -> "Open"}, Initialization :> {f[t_] := (1/2)*(1 + t/b + (1/Pi)*Sin[(t*Pi)/b])}]

Figure 5: An example of the usage of dynamical and graphical representation. The plots are prepared using *CAS Mathematica*. 
<table>
<thead>
<tr>
<th>$b$</th>
<th>$d(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.178351</td>
</tr>
<tr>
<td>0.4</td>
<td>0.155721</td>
</tr>
<tr>
<td>0.3</td>
<td>0.129224</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0974216</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0575502</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0326076</td>
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<tr>
<td>0.01</td>
<td>0.00786171</td>
</tr>
<tr>
<td>0.001</td>
<td>0.00089689</td>
</tr>
</tbody>
</table>

Table 1: Bounds for $d$ computed by (6) for various $b$.

Figure 6: The cumulative function (3) with $a=1$, $b=0.05$; Hausdorff distance $d=0.0872764$.

The Hausdorff distance $d=d(b)$ between the shifted step function and the function $f_1(t;a,b)$ satisfies the relation

$$f_1(a + d; 0, b) = 1 - \frac{1}{2} e^{-\frac{d}{b}} = 1 - d$$

or

$$d - \frac{1}{2} e^{-\frac{d}{b}} = 0.$$  \hspace{1cm} (8)

The following proposition gives upper and lower bounds for $d(b)$.

**Proposition 2.** The Hausdorff distance $d=d(b)$ between the step function $h_0$ and the $L_{cdf}(3)$ can be expressed in terms of the parameter $b$ for any real $0 < b < 1.012$ as follows:
Proof. Let us examine the function (see (9))
\[ L(d) = d - \frac{1}{2} e^{-\frac{d}{b}}. \]
From \( L'(d) > 0 \) we conclude that the function \( L \) is strictly monotonically increasing. Consider function
\[ M(d) = -\frac{1}{2} + \left(1 + \frac{1}{2b}\right)d. \]
From Taylor expansion we obtain \( M(d) - L(d) = O(d^3) \).
Hence \( M(d) \) approximates \( L(d) \) with \( d \to 0 \) as \( O(d^3) \). In addition \( M'(d) > 0 \).
Further, for \( 0 < b < 1.012 \) we have
\[ M \left( \frac{1}{3 \left(1 + \frac{1}{2b}\right)} \right) = -\frac{1}{2} + \frac{1}{3} < 0, \]
\[ M \left( \frac{\ln \left(3 \left(1 + \frac{1}{2b}\right)\right)}{3 \left(1 + \frac{1}{2b}\right)} \right) = -\frac{1}{2} + \frac{1}{3} \ln \left(3 \left(1 + \frac{1}{2b}\right)\right) > 0. \]
This completes the proof of the proposition.

Some computational examples using relation (9) are presented in Table 2 for various \( b \).

<table>
<thead>
<tr>
<th>( b )</th>
<th>( d(b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1326722</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0872764</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0286089</td>
</tr>
<tr>
<td>0.001</td>
<td>0.00467284</td>
</tr>
</tbody>
</table>

Table 2: Bounds for \( d \) computed by (9) for various \( b \) and \( a = 1 \).

4. Remarks.
1. The Semi-elliptical distribution \([14]\) is known as Wigner’s semicircle distribution. The Semi-elliptical cumulative distribution function is defined for \(-b < t < b, b > 0\) by:
\[
f_2(t; b) = \frac{1}{2} + \frac{1}{\pi} \left( \frac{t}{b} \sqrt{1 - \left(\frac{t}{b}\right)^2} + \arcsin \left(\frac{t}{b}\right) \right).
\] (11)
Proposition 3. The Hausdorff distance \( d=d(b) \) between the step function \( h_0 \) and the Semi–elliptical cumulative distribution function (11) can be expressed in terms of the parameter \( b \) for any real \( 0<b<1.28 \) as follows:

\[
\frac{1}{3 \left(1 + \frac{2}{b\pi}\right)} < d < \frac{\ln\left(3 \left(1 + \frac{2}{b\pi}\right)\right)}{3 \left(1 + \frac{2}{b\pi}\right)}.
\]

The proof of the Proposition 3 follows the ideas given in this note and will be omitted.
2. The Alpha–Skew–Laplace cumulative distribution function $f_3(t;\alpha)$ is defined by [12](see, Fig. 7):

$$f_3(t;\alpha) = \begin{cases} 
1 + \frac{(1 - \alpha t)^2}{4(1 + \alpha^2)}e^t + \frac{\alpha(1 + \alpha(1 - t))}{2(1 + \alpha^2)}e^t, & \text{if } t < 0, \\
1 - \frac{1 + (1 - \alpha t)^2}{4(1 + \alpha^2)}e^{-t} + \frac{\alpha(1 - \alpha(1 + t))}{2(1 + \alpha^2)}e^{-t}, & \text{if } t \geq 0.
\end{cases} \tag{13}$$

We note that $f_1(t;0,1) = f_3(t;0)$.

In the present paper we do not consider sigmoid functions generated as cumulative functions of other probabilistic distributions, such as the Skew–Laplace, $\alpha$–Skew Laplace, and multimodal–Skew Laplace distribution [11]–[13].

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems on his/her own.

For other results, see [6]–[13].

References


