Application Of Natural Transform To Newtonian Fluid Problems

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Abstract

This article is devoted to the applications of Natural transform to solve the following three problems of Newtonian fluid flow on an infinite plate:(i) Stokes' first problem for suddenly started and suddenly stopped plates, (ii) flow on an infinite plate, (iii) Ekman layer problem. We obtain solutions to the aforesaid problems by using the fundamental properties of Natural transform method. The concerned transform is the generalization of Laplace and Sumudu transform and so these transforms are obtained as a special cases of our considered integral transform. The plots of the concerned flow problems are also displayed.

Keywords: Stokes' first problem; Ekman layer; Natural transform; Laplace transform; Exact solutions; Newtonian Fluid.

1. Introduction

A lot of integral transforms [1- 5] are used for the solution of differential and integral equations, Laplace transform is one of the well known integral transform among them. In 1993, Watugala introduced a new version of integral transform and called it a Sumudu transform. The Sumudu transform were used to receive solutions of many dynamic and engineering problems. In 2008, Z. Khan and W. Khan[1], introduced a generalized transform known as Natural Transform . The Natural Transform was first applied to the solve some fluid flow problem according of the following form

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2},$$

subject to the initial conditions

$$u(o,y)=0,\ u(t,\infty)=0,\ \ u(t,0)=U\cos wt.$$

Then it was applied to obtain solution of Maxwell's equations [8], as given in following lines

$$\frac{\partial E_x(z,t)}{\partial z}+\mu\frac{\partial H_y(z,t)}{\partial t}=0$$

$$\frac{\partial H_y(z,t)}{\partial z}+\varepsilon\frac{\partial E_x(z,t)}{\partial t}+\sigma E_x(z,t)=0.$$

Where for a loss medium is required for the propagation of planar transverse electromagnetic wave (TEMP) in z direction with Constant permittivity ε, permeability μ and conductivity σ. Electric field vector is denoted by E while "H" indicates the magnetic field. Thereafter, various common differential equations were solved with this transforms, for example (see [9, 10]).The unsteady fluid flow problem over a plane wall was solved by using the afore said transform. At was revealed that the Natural transform converges to Laplace and Sumudu transforms. Some of the basic properties such as first shift property, change of scale property, transform of derivatives (first and second and so on) and integrals, and the table of Natural Transforms for certain functions is given in [1]. When both Laplace and Sumudu transform are changed by changing the parameters only, it give birth to Natural transform. Our considered transform plays the role of checker on the other transforms. The application of Natural transform is very useful in the solution of analytical solutions of differential and integral equations. In this article we solve, three Newtonian fluid problems by using the Natural transform for their respective analytical solutions. Maple software is used for to plot the solutions; various information can be observed from the images of the plots of the concerned problems. It is to be noted that the proposed method is very effective and simple.

2. Background Materials

This section is devoted to the some fundamental results which are needed in this article.

Definition 2.1: Let $f(t)$ be a function defined for all $t\geq 0$. Then Natural transform of $f(t)$ is the function $R(s,u)$ defined by

$$R(s,u)=\int_0^{\infty} f(ut)e^{-st}dt\ , \tag{1}$$

provide the integral on the right side converges. Where $s,u\in(-\alpha,\alpha)$ are the parameters of the transform, and transform is defined over the set of function given by

$$A=\{f(t): there\,exist\ M,\alpha_1,\alpha_2>0, |f(t)|<Me\frac{|\alpha_1|}{\alpha_2}, if\ t\in(-1)^j\times[0,\infty)\}. \tag{2}$$

3. Main results and Discussion

Stokes's first problem

Assume thatr a cartesian coordinate system is taken, x-axis is situated along an infinitely long flat plate, the half-space is occupied by an incompressible viscous fluid $y\geq 0$, as the fluid is viscous, the plate's effect would diffuses into the fluid. In case of motion of the boundary is in the x-direction, it is assumed that motion of the fluid would be in the same direction. As a result, the only non-zero velocity component will be U and it will be a function of y and t only. Therefore

$$U=U(y,t), V=0, W=0\ . \tag{3}$$

As " U" is independent of X , the pressure will be independent of y , thus p will be independent of x, it means that the pressure will be constant everywhere in the fluid. Using such properties of the flow field, the governing equations are reduced to the linear partial differential equations as in [7, 12], and provided by

$$\frac{d}{dt}U(y,t)=v\frac{d^2U(y,t)}{dy^2} \quad . \tag{4}$$

Suddenly started plate

Both the plate and the fluid are static at the beginning, but the plate starts motion at once with a constant velocity U_0 and continue the motion with this velocity for $t>0$. The motion of the plate will be communicated to the fluid because it is viscous.

$$U(0,t)=\begin{cases}0 & for \quad t\le 0, \qquad (5)\\ U_0 & for \quad t>0, \qquad (6)\end{cases}$$

$$U(\infty,t)=0.$$

The problem is well posed because we have governing equations, initial condition, and boundary condition. The two variables are reduced to a single variable using the Natural transform method.

Therefore, partial differential equations are transferred to ordinary differential equations; Thus inview of the difficulties associated with the original differential equation will be reduced. A set comprised of governing equations together with initial conditions and boundary conditions could be solved by Natural transform technique. By using the considered transform, Eq(4) and the boundary conditions take the following forms

$$\frac{d^2\overline{U}(y,t)}{dy^2}-\frac{s}{vu}\overline{U}(y,s,u)=0 \tag{7}$$

and

$$\left.\begin{aligned}\overline{U}(0,s,u)=U_0\\ \overline{U}(\infty,s,u)=0\end{aligned}\right\}. \tag{8}$$

The general solution to Eq.(7) is

$$\overline{U}(y,s,u)=C_1e^{\sqrt{\frac{s}{vu}}y}+C_2e^{-\sqrt{\frac{s}{vu}}y} \quad . \tag{9}$$

By using boundary conditions (8) for solving the arbitrary constants C_1 and C_2, and substituting the values of these constants into Eq (9), we receive

$$\overline{U}(y,s,u)=U_0e^{-\sqrt{\frac{s}{vu}}y} \quad . \tag{10}$$

Taking inverse Natural transform of Eq(10), the velocity becomes

$$U(y,t)=U_0N^{-1}\left\{e^{-\sqrt{\frac{s}{vu}}y}\right\}$$

$$U(y,t)=U_0\left[1-erf\left(\frac{y}{2\sqrt{vt}}\right)\right]=U_0erfc\left(\frac{y}{2\sqrt{vt}}\right), \tag{11}$$

where $$erf(\eta)=\frac{2}{\sqrt{\pi}}\int_0^{\eta}e^{-\xi^2}d\xi . \tag{12}$$

is the complementary error function. The values of the complementary error function are tabulated in Table 1. We have plotted Velocity by assuming the values of $\mu = 0.61, \nu = 1.5\times10^{-5}$ m 2 / s cm in Figure 1.

η	$erfc(\eta)$	H	$erfc(\eta)$
0	1.0	1.1	0.11980
0.05	0.94363	1.2	0.08969
0.1	0.88754	1.3	0.06599
0.15	0.83200	1.4	0.04772
0.2	0.77730	1.5	0.03390
0.25	0.72367	1.6	0.02365
0.3	0.67137	1.7	0.01621
0.35	0.62062	1.8	0.01091
0.4	0.57161	1.9	0.00721
0.5	0.47950	2.0	0.00468
0.6	0.39615	2.5	0.000407
0.7	0.32220	3.0	0.0000221
0.8	0.25790	3.5	0.00000074
0.9	0.20309	4.0	0.00000001
1.0	0.15730	∞	0.0

Table .1. The tabulated values of the complementary error function for different values of η.

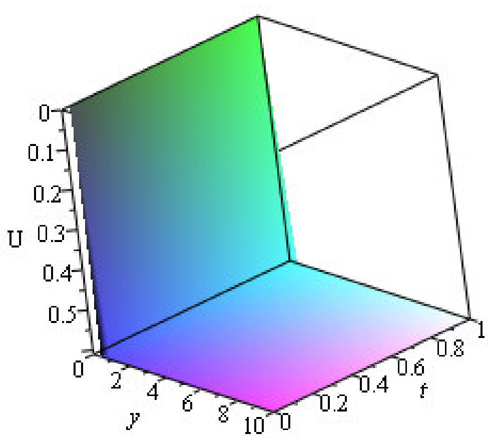

Figure 1. Plot of velocity corresponding to time and position.

Suddenly stopped plate

Both the fluid and the plate will move with a uniform speed U_0 at the initial stages; after a short time while the plate is decelerated to zero velocity; thus the boundary conditions are

$$U(0,t) = \begin{cases} U_0 & for \quad t \le o, \\ 0 & for \quad t > 0, \end{cases} \quad , \tag{13}$$

$$U(\infty, 0) = 0 . \tag{14}$$

Taking Natural transform of Eq.(4) and the boundary conditions and using initial condition, we get

$$\frac{d^2\overline{U}}{dy^2}-\frac{s}{vu}\overline{U}(y,s,u)=-\frac{U_0}{sv} \quad (15)$$

$$\left.\begin{array}{l}\overline{U}(0,s,u)=U_0\\ \overline{U}(\infty,s,u)=0\end{array}\right\}, \quad (16)$$

the general solution of the non-homogeneous differential Eq.(15) is

$$\overline{U}=C_1e^{\sqrt{\frac{s}{vu}}y}+C_2e^{-\sqrt{\frac{s}{vu}}y}+\frac{u}{s}U_0 \ . \quad (17)$$

Using boundary conditions (16), we obtain

$$\overline{U}(y,s,u)=\frac{U_0}{s}\left(1-e^{-\sqrt{\frac{s}{vu}}y}\right) \quad (18)$$

$$U(y,t)=U_0 erf\left(\frac{y}{2\sqrt{vt}}\right). \quad (19)$$

It could be noted that the solutions given by Eq(11) and (19) are similer with that of Laplace transforms and similarity methods [6,7,12]. It is clear from the above mentioned Eqs (11) and (19) that in both the cases, the plate's effect diffuses into the fluid at a rate that is proportional to the square root of the kinematics viscosity. Usually the shear layer thickness is defined as the point where the wall effect on the fluid has dropped to 1 percent in the first case (suddenly started plate), where $U/U0 = 0.01$, in the second case (suddenly stopped plate), where $U/U0 = 0.99$. These both correspond to erf(η) = 0.01, or η = 1.82. Then the shear layer thickness δ in these flows is approximately as given

$$\delta \approx 3.64\ \nu t. \quad (20)$$

For example, for air at C20° with , ν = 1.5E − 5 m 2 / s cm , δ ≈ 11 after 1 min.

Flow on an infinite plate

Imagine the flow of a viscous fluid on an infinite plate provided the pressure is constant. If x-axis is the direction of main flow, then $U \neq 0, V=0, W=0$. The continuity equation is

$$\frac{du}{dx}=0 .$$

Y-axis is taken normal to the plate, then it could be concluded that $U=U(y)$. The equations of motion are reduce to one equation which is [8]

$$\frac{d^2U}{dx^2}=0, p=\text{Constant}. \quad (21)$$

Taking no-slip condition

$$U=0 \ at \ y=0 , \quad (22)$$

and denoting the wall shear as

$$\tau_w=\mu\left(\frac{dU}{dy}\right)_{y=0} . \quad (23)$$

The Natural transform of Eq.(21) and use the conditions (22) and (23) yields

$$\overline{U}(u)=\frac{\tau_w}{\mu}\frac{u}{s^2} \ . \quad (24)$$

And the inverse Natural transform of the above equation gives

$$U(y)=\frac{\tau_w}{\mu}y \ . \qquad (25)$$

Eq(25) reveals that the distribution of velocity is linear in y. This result is similer with the analytical solution obtained in [8].

Velocity U is plotted in the Figure 2, to the corresponding values at different values of shear stress $\tau = 1, 2, 3$ and for position taking the values $y = 0, ..., 10$. The plot indicates that the flow of fluid is dependent on wall shear stress and positions as displayed in the Figure 2.

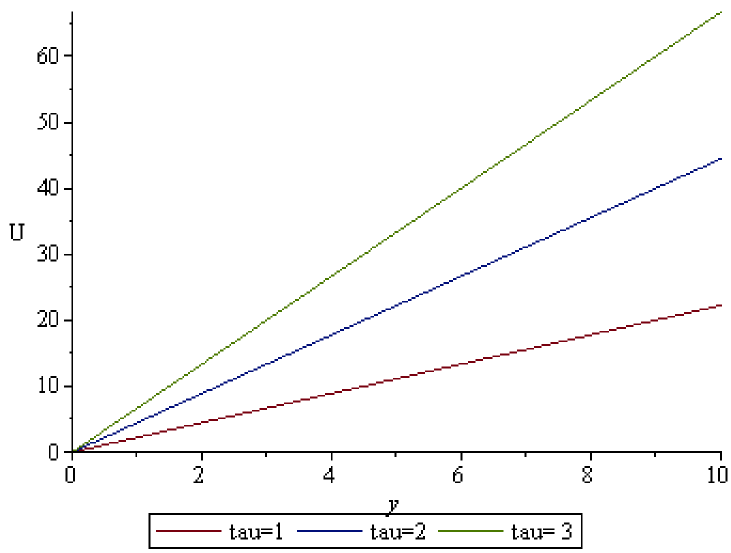

Figure 2. Plot of velocity against position and fixed values of wall shear stress

Ekman layer problem

Think upon a viscous fluid on a surface, the latter is in a rotatary motion with a constant angular velocity Ω; the surface is considered to be almost flat and fluid to have a horizontal free surface; the rectangular cartesian system is introduced on the free surface with $Z = 0$ at the free surface z-axis along the normal to the free surface. Let us suppose the free surface is subject to a constant shearing force μS on the x-axis. As the motion is uniform, steady, and unobstructed

$$U = U(z), V = V(z), W = 0 \ . \qquad (26)$$

Further by absorbing the centripetal and the body forces, a modified pressure P as in [8] is defined as

$$P = p + \rho\chi - \frac{1}{2}\rho\Omega^2 R^2 \qquad (27)$$

$$\frac{dP}{dx} = 0, \frac{dP}{dy} = 0 \qquad (28)$$

$$-2\Omega_3 V = v\frac{d^2U}{dz^2} \qquad (29)$$

$$2\Omega_3 U = v\frac{d^2V}{dz^2} \ . \qquad (30)$$

The pressure gradient in the z-direction is balanced by the Carioles force as in [8],

$$2\left(\Omega_1 V - \Omega_2 U\right) = -\frac{1}{\rho}\frac{dP}{dz}. \quad (31)$$

Note that $\Omega_3 = \Omega\cos\theta$, where θ is the angle between the vector Ω and the unit vector k along the z-axis. Taking Natural transform of Eq.(29), (30) and using the following conditions

$$U_0 = \frac{S}{2K}$$

$$U^{/}(0) = S \quad (32)$$

$$K = \left(\Omega_3 / v\right)^{\frac{1}{2}} . \quad (33)$$

We obtain

$$-2\Omega_3 \overline{V}(s,u) = v\left[\frac{s^2\overline{U}(s,u)}{u^2} - \frac{s}{u^2}.\frac{S}{2K} - \frac{S}{u}\right], \quad (34)$$

$$2\Omega_3 \overline{U}(s,u) = v\left[\frac{s^2}{u^2}\overline{V}(s,u) + \frac{S.s}{2Ku^2}\right] \quad (35)$$

Coupling equations (34) and (35), we get

$$\left(2i\Omega_3 u^2 - v\right)\lambda(u) = -v\left(\frac{S}{2K}(1-i) + su\right). \quad (36)$$

Where

$$\lambda(u) = \overline{U} + i\overline{V} , \quad (37)$$

$$\overline{U}(s,u) = \frac{S.s^3}{2K\left(1+4k^4u^4\right)} + \frac{S.\frac{u}{s}.s^3}{1+4k^4u^4} + \frac{kS.\frac{u^2}{s^2}.s^3}{1+4k^4u^4}, \quad (38)$$

$$\overline{V}(s,u) = \frac{S.s^3}{2K\left(1+4k^4u^4\right)} + \frac{kS.\frac{u^2}{s^2}.s^3}{1+4k^4u^4} + \frac{2k^2S.\frac{u^3}{s^3}.s^3}{1+4k^4u^4} . \quad (39)$$

Taking Natural transform of Eq (38) and (39) and simplifying the resulting equations to obtain

$$U(z) = \frac{S}{\sqrt{2k}} e^{kz}\cos\left(kz - \frac{\pi}{4}\right) , \quad (40)$$

$$V(z) = \frac{S}{\sqrt{2k}} e^{kz}\sin\left(kz - \frac{\pi}{4}\right). \quad (41$$

It is worth mentioning again that these solutions are identical solutions [8].

At a depth $z = \frac{\pi}{k} or (z = -\pi / k)$ the velocity vector is $V = -\frac{S}{2k}e^{-\pi}(i-j)$.

Thus at a depth $\frac{\pi}{k}$ the velocity vector has decreased by a factor of $e^{-\pi}$ and its direction has become opposite to that at the free surface. The depth S=1; $\pi / k = \pi(\upsilon / \Omega_3)^{\frac{1}{2}}$ is a measure of the Ekman layer thickness. Next we plot the graph of velocity at the defined parameters as in Figures 3, 4, 5 respectively.

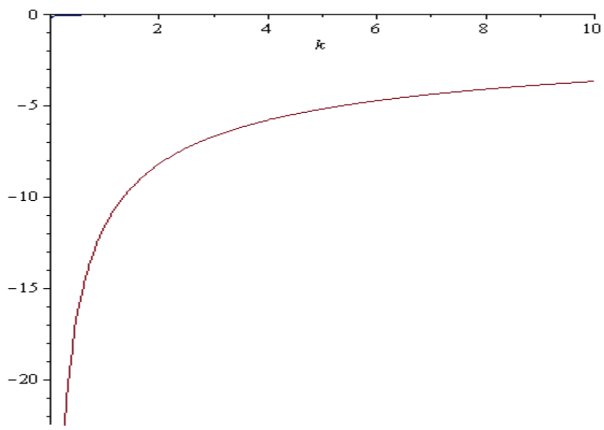

Figure 3. Velocity graph of U at specified value of $z = \frac{-\pi}{k}$

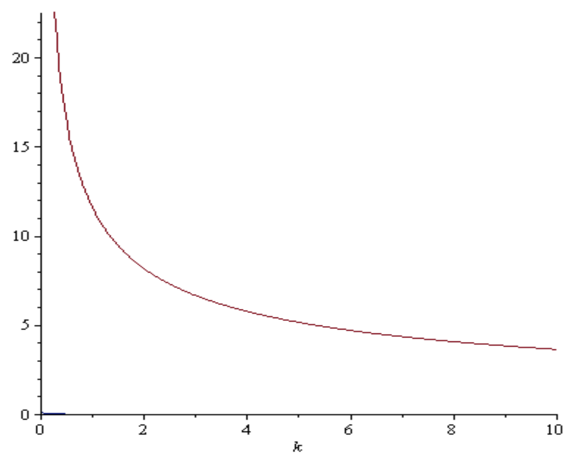

Figure 4. Velocity graph of U at specified value of $z = \frac{\pi}{k}$**.**

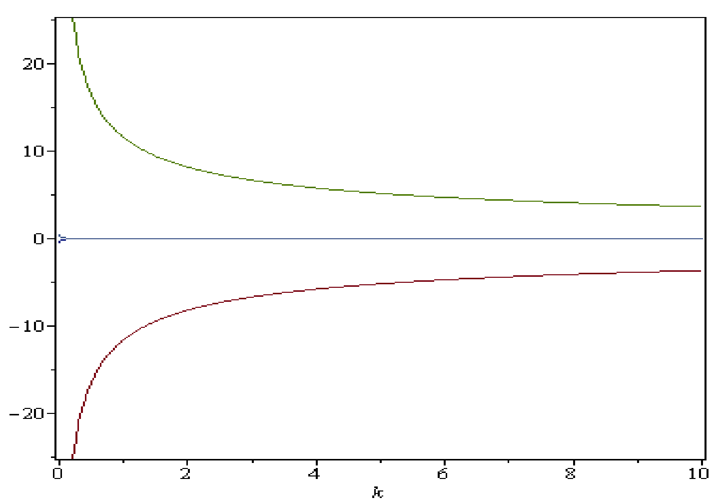

Figure 5. Plot of velocity U and V for $Z = \frac{-\pi}{K}$ **and** $Z = \frac{\pi}{k}$ **.**

Conclusion

In this article, we successfully applied Natural transform to solve three Newtonian fluid problems. The results are identical to those given in the literature obtained by other methods such as by Laplace and Sumudu transform. Thus Natural transform is the strongest tool as it. It converges to both Laplace and Sumudu transform. The results obtained by using Natural transform are similer to that obtained by other method and transform like similarity method and integral transform method.

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