Sphere's Volume-Area Formula

Kern E. Kenyon

4632 North Lane, Del Mar
CA, 92014, USA
Email: kernken@aol.com

Abstract
The volume-area formula for a sphere is displayed: volume equals one third the area times the radius. In order to understand the correctness of this formula it is proposed to fill the sphere with pyramids such that all the vertices are located at the sphere's center and all the corners of the bases touch the sphere's surface. Triangular bases are selected for discussion. By increasing the number of pyramids indefinitely, and keeping the standard formula for a pyramid in mind, it is evident that the sphere's volume-area formula stated above should be exact.

Key Words: Sphere, Volume-Area Formula

Introduction
In equation (1) is the volume-area formula for a sphere

$$V = \frac{1}{3}Ar \tag{1}$$

Where V is volume, A is surface area and r is radius. Whether or not this equation has ever been put down in print before is not known to me but it is not particularly important for the present discussion. One kind of verification comes immediately from the separate well-known formulae for the volume and the area of a sphere:

$V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

Which are assumed to be exactly true in what follows. Notice that in equation (1) π does not appear explicitly. Also it is not intuitive that the volume of a sphere should be proportional to the product of the area and the radius in the first place and with the proportionality factor being simply 1/3.
What is proposed here is possibly a new method of understanding the correctness of equation (1). This method was stimulated by extrapolating from the available procedure of arriving at the ancient formula for area of a circle, $A=(1/2)Cr$, where C is the circumference, by starting with a regular polygon and increasing

the number of sides indefinitely [1]. This procedure has been known for a long time and makes use of the fact that the area of a regular polygon equals one half the perimeter times the shortest distance between the polygon's center and the mid-point of one side.

Method

Begin by considering that quite a few radii are more or less randomly extending from the center of a sphere to its surface. Where the radii touch the surface connect straight lines to form triangles, each of which will serve as the base of a pyramid. Triangular bases are not necessary but just simpler to think about than bases with more sides than three. Equilateral triangles are not mandatory for the bases. Now the well-known volume of a pyramid is one third the area of the base times the altitude. Consequently an approximation of the volume of the sphere is obtained by adding up the volumes of all the pyramids. Already it is seen where the factor of one third in equation (1) probably comes from.

At any stage, including the initial one, the altitudes of the pyramids are smaller than the radius of the sphere and the areas of the bases are smaller than the pieces of the surfaces of the sphere "above" them. Thus the volume obtained by adding up the volumes of all the pyramids will be smaller than given by equation (1) because both the area computed by adding all the base areas will be smaller than that given by (1) and all the altitudes of the pyramids will be smaller than the radius. However, more radii from the sphere's center can be created and more triangular bases made causing the altitudes of the corresponding pyramids to become closer to equaling the radius and the areas of the now smaller bases become closer to equaling the pieces of the surface of the sphere above them. When this process is carried out indefinitely, evidently the result will converge to equation (1).

By the above chain of reasoning equation (1) can be understood to be exact. Such a logical deduction is a direct analogue of that used to verify the ancient formula for the area of a circle mentioned earlier.

Discussion

One difference between the two and three dimensional problems is as follows. In the method of understanding the area of a circle a regular polygon is adopted. But when a sphere is filled with triangular based pyramids, they may not turn out to be all identical. From one pyramid to another slight differences can occur in their altitudes and in the areas of their bases. However, that will not affect the final result because as the number of pyramids steadily increases all their respective altitudes will converge to be equal to each other and to the radius of the sphere. Similarly, all the areas of the bases will converge to be equal to each other and to pieces of the surface of a sphere.

Consequently, it is possible to conjecture that it was not necessary to use a regular polygon in order to demonstrate the correctness of the area formula for the circle (1/2 the circumference times the radius).

Reference

[1] Newman, J. R. (1956), The World of Mathematics, Simon and Schuster, New York, Volume 1, page 40.