ON FUZZY NORMED SPACES CATEGORY

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Abstract:
In this paper after an introduction of category theory, fuzzy normed linear spaces will be introduced. We introduce the notions of fuzzy normed linear spaces category and discuss a relationship between fuzzy normed linear spaces category and linear spaces category. Also we give some definitions and theorems about fuzzy normed linear spaces category.

Keywords: Category Theory, Fuzzy Normed Space, Fuzzy Theory

1. Introduction:

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. Fuzzy sets were introduced by Lot A. Zadeh [6] in 1965 as an extension of the classical notion of set. Zadeh's extension principle provides a mathematical approach for extending classical functions to fuzzy mappings. It has been considered that extension principle is an important tool in the development of fuzzy arithmetic and other areas.

Since its inception in 1965, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Mathematical developments have advanced to a very high standard and are still forthcoming to day.

The concept of a fuzzy norm on a linear space is of comparatively recent origin. It was Katsaras, who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. Following his pioneering work, Felbin offered in 1992 an alternative definition of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated this fuzzy norm is of the Kaleva and Seikkala type. A further development along this line of inquiry took place when in 1994, Cheng and Mordeson evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek type in 2003 Xiao and Zhu redefined, in a more general setting, the idea of Felbin's definition of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space.

Following Cheng and Mordeson, in 2003 Bag and Samanta have introduced a definition of a fuzzy norm whose associated metric is similar to Kramosil and Michelek type metric. The novelty of their approach is that they have been able to prove a decomposition theorem of fuzzy norms into a family of crisp norms and using this they have studied some properties of finite dimensional fuzzy normed linear spaces. After this paper they made several papers about fuzzy normed spaces.

In our study using Bag and Samanta type fuzzy norm, we discuss a relationship between fuzzy normed linear spaces category and linear spaces category.

2. Preliminaries:

For each mathematical discipline we define at first objects and then admissible maps for describing the objects. This procedure is formalized by the concept 'category'. Now, some basic and fundamental notions related to the subject of the paper are introduced.

2.1. Definition: [4]

A category C consists of

1) a class $ob(C)$ of objects (which denoted by $A, B, C, ...$)

2) A class of pairwise disjoint sets $[A, B]_C$ for each pair $(A, B)$of objects (the mem-bers of $[A, B]_C$ are called morphisms from A to B), and

3) a composition of morphisms, i.e. for each triple (A, B, C) of objects there is a map.

$$[A,B]_C \underset{(f,g)}{\times} [B,C]_C \underset{\rightarrow}{\rightarrow} \underset{g\circ f}{[A,C]_C}$$

(where x denotes the cartesian product) such that the following axioms are satisfied:

cat$_1$) (Associativity). If $f \in [A,B]_C$, $g \in [B,C]_C$ and $h \in [C,D]_C$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

cat$_2$) (Existence of identities). For each $A \in ob(C)$ there is an identity (morphism) $1_A \in [A,A]_C$ such that for all $B, C \in ob(C)$,all $f \in [A,B]_C$ and all $g \in [C,A]_C$, $f \circ 1_A = f$ and $1_A \circ g = g$.

2.1. Remark:

1) We write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in [A,B]_C$. A (resp B) is called the domain of f (resp. the codain of f).

2) a) The identity 1_A is uniquely determined by A.

b) If $A, A' \in ob(C)$ with $A \neq A'$ then, $1_A \neq 1_{A'}$, because $[A,A]_C \cap [A',A']_C = \emptyset$.

3) The class of all morphisms of C is denoted by

$$MorC := \bigcup_{(A,B)\in ob(C)\times ob(C)} [A,B]_C;$$

its elements are called C-morphisms.

2.1. Examples:

1) The category $\underline{Set}$ of sets and maps: $ob(\underline{Set})$ is the class of all sets; $[A, B]_{\underline{Set}}$ is the set of all maps from A to B for all $\forall A, B \in ob(\underline{Set})$. The composition of morphisms is the usual composition maps.

2) The category $\underline{Mod}_R$ of R-modules and R-linear maps (where R denotes a commutative ring with unit): $ob(\underline{Mod}_R)$ is the class of all R-modules and Mor $\underline{Mod}_R$ is the class of all R-linear maps (between any two R-modules). The composition of morphisms is the usual composition of maps.

3) The category $\underline{Top}$ of topological spaces and continuous maps.

2.2. Definition: [4]

Let C be a category and $f \in [A, B]_C$ with $(A, B) \in ob(C) \times ob(C)$. Then f is called an isomorphism provided that there is some $g \in [B, A]_C$ with $g \circ f = 1_A$ and $f \circ g = 1_B$. If $f \in [A, B]_C$ is an isomorphism then A and B are called isomorphic (denoted by $A \cong B$)

2.2. Remark:

1) Afornamed g is uniquely determined by f (if $g' \in [B, A]_C$ with $g' \circ f = 1_A$ and $f \circ g' = 1_B$ then $g = g \circ 1_B = g \circ (f \circ g') = (g \circ f) \circ g' = 1_A \circ g' = g'$) and is denoted by f^{-1}.

2) Obviously, an isomorphism in $\underline{Set}$ is a bijective map (and vice versa), while an isomorphism in $\underline{Top}$ is a homeomorphism (and vice versa).

3) For every category C, the identity $1_X : X \to X$ is an isomorphism for each

$X \in ob(C)$. If $f : X \to Y$ is an isomorphism in C, then $f^{-1} : Y \to X$ is also an isomorphism. Additionally, the composition of two isomorphisms in C is again an isomorphism.

As well-known mathematical objects may be described by means of maps. There is an analogous description of categories via so-called functors. The classical definition of universal maps in the sense of N.Bourbaki corresponds to a categorical one with respect to a functor.

2.3. Definition: [4]

Let C and D be categories, let $F_1 : ob(C) \to ob(D)$ and $F_2 : MorC \to MorD$ be maps. Instead of $F_1(A)$ we write $F(A)$ and instead of $F_2(f)$ we write $F(f)$. Then $F := (C, D, F_1, F_2)$ is called a functor from C to D or more exactly a covariant functor (denoted by $F : C \to D$) provided the following are satisfied:

F_1) $f \in [A, B]_C$ implies $F(f) \in [F(A), F(B)]_D$

F_2) $F(f \circ g) = F(f) \circ F(g)$, provided $f \circ g$ is defined (i.e. the domain of f is equal to the codomain of g)

F_3) $F(1_A) = 1_{F(A)}$. $(A \in ob(C))$

If F_1) and F_2) are replaced by

$F_1^{'}$) $f \in [A, B]_C$ implies $F(f) \in [F(B), F(A)]_D$

$F_2^{'}$) $F(f \circ g) = F(g) \circ F(f)$ (provided $f \circ g$ is defined in C)

respectively, then F is called a contravariant functor from C to D.

2.2. Examples:

1) The identity fuctor: $I : C \to C$ maps objects and morphisms identically to themselves (covariant functor).

2) Constant functors: Let C and D be arbitrary categories, let $X \in ob(D)$. For every $A \in ob(C)$ and every $f \in MorC$, put $F(A) = X$ and $F(f) = 1_X$ (co- and contravariant functor).

3) Forgetful (or underlying) functors: Let C be a topological category, $\underline{Set}$ be the category of sets and maps and let $F : C \to \underline{Set}$ be defined by $F((X, \tau)) = X$ and $F(f) = f$ (=map between the underlying sets) (covariant functor).

4) Inclusion functors: Let C be a category, A be a subcategory of C, i.e. A is a category such that

a) $ob(A) \subset ob(C)$

b) $[A, B]_A \subset [A, B]_C$ for each $(A, B) \in ob(A) \times ob(A)$

c) The composition of morphisms in A coincides with the composition of these morphisms in C.

d) For each $A \in ob(A)$ the identity 1_A is the same in A and C.

(If $[A, B]_A = [A, B]_C$ is satisfied for each $(A, B) \in ob(A) \times ob(A)$ instead of b), then A is called full.)

The inclusion functor $F_e : A \to C$ is defined by $F_e(A) = A$ for each $A \in ob(A)$ and$F_e(f) = f$ for each $f \in MorA$ (covariant functor).

2.4. Definition: [6]

Let $X \neq \emptyset$ be a set and $I = [0,1] \subset R$ be a closed interval. A fuzzy set A in X is a set of ordered pairs $\ni$ $A = \{(x, A(x)) : x \in X\} \subset X \times I$,which is characterized by its membership function $A : X \to I$. $A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

2.3. Example:

The objects of category of fuzzy subsets $ob(F) = ob(\underline{Set}(\overline{[0,1]}))$ are all pairs (X, A) where X is a set and $A : X \to I$ is a function from X to the unit interval. The maps of F are defined by $MorF = \bigcup_{((X,A),(Y,B)) \in ob(F) \times ob(F)} [(X.A),(Y,B)]_F = \{f \in [X,Y]_{\underline{Set}} : A(x) \leq B(f(x)), \forall x \in X\ \}$ where $\underline{Set}$ denotes the category of sets. This collection of maps is closed under composition, and with composition of function, F is a category. It is called fuzzy subset category.

2.3. Remark:

There is a forgetful functor $U : F \to \underline{Set}$, namely $U(X, A) = X$, $U(f) = f$.

2.4. Remarks: [2]

1) Let $(R, +, \cdot)$ be a commutative ring with identity, for each $t \in [0,1]$, the set $A_t = \{x \in R : A(x) \geq t\}$ is called a level subset of R and $A = B$ if and only if $A_t = B_t$the set $A^* = \{x \in R : A(x) > 0\}$ is called the support of R.

2) Let $x \in X$ and $t \in [0,1]$, let x_t denote the fuzzy subset of X defined by $x_t(y) = 0$ if $x \neq y$ and $x_t(y) = t$ if $x = y$ for all $y \in R$. x_t is called a fuzzy singleton. If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x+y)_\lambda$ and $x_t \circ y_s = (x.y)_\lambda$, where $\lambda = \min\{t, s\}$.

3) Let $I^R = \{A_i : i \in \Lambda\}$ be a collection of fuzzy subset of R. Define the fuzzy subset of R (intersection) by $(\bigcap_{i\in\Lambda} A)(x) = \inf\{A_i(x) : i \in \Lambda\}$ for all $x \in R$. Define the fuzzy subset of R (union) by $(\bigcup_{i\in\Lambda} A)(x) = \sup\{A_i(x) : i \in \Lambda\}$ for all $x \in R$.

3. Fuzzy Normed Linear Spaces Category:

In this section we studied and discussed the concept of fuzzy normed linear spaces. We gave some definitions, examples and theorems about fuzzy norm, fuzzy linear transformation, fuzzy boundednes, fuzzy continuity and etc.. We gave a new proposition about the functor from fuzzy normed linear spaces category to vector spaces category.

3.1. Definition: [2]

A vector space (linear space) over a field F is a set U, whose elements are called vectors which two operations, additon $(+ : U \times U \to U)$ and scalar multiplication $(\cdot : F \times U \to U)$ with conditions is satisfies,

$\forall x, y, z \in U$, $\forall \lambda, \mu \in F$ için;

1) $x + y \in U$

2) $x + y = y + x$

3) $(x + y) + z = x + (y + z)$

4) There exist $\exists \theta \in U$ such that $x + \theta = x$

5) $\exists - x \in X$ such that $x + (-x) = \theta$

6) $\lambda x \in U$

7) $1 \cdot x = x$

8) $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$

9) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$

10) $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu.) \cdot x$

3.2. Definition: [5]

A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if $*$ satisfies the following conditions:

i. $*$ is commutative

ii. $*$ is continuous

iii. $a * 1 = 1$, $\forall a \in [0, 1]$

iv. $a * b \leq c * d$, where $a \leq c, b \leq d$ and $a, b, c, d \in [0, 1]$

3.3. Definition: [5]

Let U be real vector space, let $\|.\|$ be real-valued function on U satisfying the following conditions,

1) $\|x\| \geq 0$

2) $\|x\| = 0$ if and only if $x = 0$

3) $\|\alpha x\| = |\alpha| \|x\|$, where α is real

4)$\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Here $x, y \in U$, $\|.\|$ is called a norm on U and $(U, \|.\|)$ is called a linear normed space.

3.4. Definition: [1]

Let U be a linear space over a field F (field of real or complex numbers). A fuzzy subset N of UxR is called a fuzzy norm on U iff $\forall x, u \in U$ and $c \in F$,

N_1) $\forall t \in R$ with $t \leq 0$, $N(x, t) = 0$,

N_2) $\forall t \in R$, $t > 0$, $N(x, t) = 1 \Leftrightarrow x = 0$,

N_3) $\forall t \in R$, $t > 0$, $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$,

N_4) $\forall t, s \in R$, $x, u \in U$; $N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$,(where the binary operation $*$ is taken to be the minimum.)

N_5) $N(x, .)$ is a non-decreasing, left continuous function of R and $\lim_{t \to \infty} N(x, t) = 1$.

The pair (U, N) will be referred to as a fuzzy normed linear space.

N is a fuzzy norm on U in the sense that associated to $x \in U$ and $t \in R$, $N(x, t)$ indicates the truth value statement "the real number t is the norm of x" and which belongs to $[0, 1]$.

3.1. Example: [1]

Let $(U, \|\|)$ be a normed linear space. Define, for $\forall x \in U$ and $t \in R$

$$N(x,t) = \begin{cases} \dfrac{t}{t + \|x\|} & ; t > 0 \\ 0 & ; t \leq 0 \end{cases}$$

Then (U, N) is a fuzzy normed linear space.

3.2. Example: [1]

Let $(U, \|\|)$ be a normed linear space. Define, for $\forall x \in U$ and $t \in R$

$$N(x,t) = \begin{cases} 0 & ; t \leq \|x\| \\ 1 & ; t > \|x\| \end{cases}$$

Then (U, N) is a fuzzy normed linear space.

3.5. Definition: [5]

Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces. $T : U \to V$ be an operator. T is said to be fuzzy bounded if and only if there exist $h > 0$ and $0 < r < 1$, such that: $N_2(T(x), h) > rN_1(x, h)$.

3.6. Definition: [5]

Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces. A mapping $T : U \to V$ is said to be fuzzy continuous at $x_0 \in U$, if for every $\varepsilon \in (0, 1)$, $t > 0$ there exist $\delta \in (0, 1)$, $s > 0$ such that $N_1(x - x_0, s) > 1 - \delta$ implies that $N_2(T(x) - T(x_0), t) > 1 - \varepsilon$, for all $x \in U$.

Then T is continuous on U if it is fuzzy continuous at each point of U.

3.7. Definition: [5]

An operator T from (U, N_1) to (V, N_2) is said to be strong fuzzy continuous at $x_0 \in U$, if for given $t > 0$, there exist $s > 0$, such that for all $x \in U$; $N_2(T(x) - T(x_0), t) \geq N_1(x - x_0, s)$. T is said to be strongly fuzzy continuous on U if T is strong at each point of U.

3.8. Definition: [5]

Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces and $T : (U, N_1) \to (V, N_2)$ be an operator. For $\forall x, y \in U$,
$\in \mathbb{R}$;

if $T(x+y) = T(x)+T(y)$ and $T(ax) = aT(x)$ be provide, then T is a linear operator.

3.1. Theorem: [5]

Let $T : (U, N_1) \to (V, N_2)$ be a linear operator, then T is fuzzy continuous if and only if T is fuzzy bounded.

Proof:

We consider two cases for T.
For $T = 0$, the statement is trivial.

Let $T \neq 0$, then $\|T\| \neq 0$, and we will assume that T is fuzzy bounded and consider any $x_0 \in U$.

Suppose for any given $t > 0$, $\varepsilon \in (0,1)$; there exists $s > 0$, $\delta \in (0,1)$.

Since T is linear for every $x \in U$, such that:

$$\begin{aligned} N_1(x - x_0, s) &> 1 - \delta \Rightarrow \frac{s}{s + \|x - x_0\|} > 1 - \delta \\ &\Rightarrow \|x - x_0\| < \frac{s\delta}{1 - \delta} \end{aligned}$$

Let $\dfrac{s\delta}{1-\delta} = \dfrac{t\varepsilon}{\|T\|(1-\varepsilon)}$

$$\begin{aligned} \|T(x) - T(x_0)\| &= \|T(x - x_0)\| \\ &\leq \|T\| \|x - x_0\| \\ &< \|T\| \frac{t\varepsilon}{\|T\|(1-\varepsilon)} \\ &= \frac{t\varepsilon}{(1-\varepsilon)} \end{aligned}$$

$\Rightarrow \|T(x) - T(x_0)\| < \dfrac{t\varepsilon}{(1-\varepsilon)}$

$\Rightarrow \|T(x) - T(x_0)\|(1-\varepsilon) + t - t - t\varepsilon < 0$

$\Rightarrow \|T(x) - T(x_0)\|(1-\varepsilon) - t + t(1-\varepsilon) < 0$

$\Rightarrow (1-\varepsilon)(\|T(x) - T(x_0)\| + t) < t$

$$\Rightarrow \frac{t}{t + \|T(x) - T(x_0)\|} > (1 - \varepsilon)$$

$$\Rightarrow N_2(T(x) - T(x_0), t) > (1 - \varepsilon)$$

Since $x_0 \in U$ was arbitrary. This shows that T is continuous.

Conversely, assume that T is continuous at an arbitrary $x_0 \in U$. Then given $t > 0$, $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$, $s > 0$, such that:

$$N_1(x - x_0, s) > 1 - \delta \Rightarrow N_2(T(x) - T(x_0), t) > 1 - \varepsilon$$

We now take any $y \neq 0 \in (U, N_1)$ and set:

$$x = x_0 + \frac{s\delta}{(1 - \delta)\|y\|} y \Rightarrow x - x_0 = \frac{s\delta}{(1 - \delta)\|y\|} y$$

Since T is continuous, then:

$$N_2(T(x) - T(x_0), t) > 1 - \varepsilon$$

$\Rightarrow$

$$\begin{aligned}
\frac{t}{t + \|T(x) - T(x_0)\|} &= \frac{t}{t + \|T(x - x_0)\|} \\
&= \frac{t}{t + \left\|T(\frac{s\delta}{(1 - \delta)\|y\|} y)\right\|} \\
&= \frac{t}{t + \|T(y)\| \frac{s\delta}{(1 - \delta)\|y\|}}
\end{aligned}$$

$$\Rightarrow \|T(y)\| \frac{s\delta}{(1 - \delta)\|y\|} < \frac{t\varepsilon}{(1 - \varepsilon)}$$

$$\Rightarrow \|T(y)\| < \frac{t\varepsilon(1-\delta)}{(1-\varepsilon)s\delta}\|y\|$$

Let $r = \dfrac{t\varepsilon(1-\delta)}{(1-\varepsilon)s\delta}$

Hence $\|T(y)\| < r\,\|y\|$, such that there exist $h > 0$ and we can write:

$$\frac{h}{h + \|T(y)\|} > r\frac{h}{h + \|y\|}$$

$$\Rightarrow N_2(T(y), h) > rN_1(y, h)$$

$\Rightarrow T$ is fuzzy bounded operator.

3.2. **Theorem:**

Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces and let $T : U \to V$ be a linear function. Then T is fuzzy continuous at every point of U.

3.9. **Definition:** [2]

Let A, B be two fuzzy subspaces of vector spaces U, V over F respectively. $K : A \to B$ is called a fuzzy linear transformation on fuzzy subspace if

$$K(\lambda x_t, \alpha y_h) \geq \min\{K(x_t), K(y_h)\}$$

for all $x_t, y_h \in A$, $\lambda, \alpha \in F$ and $t, h \in [0, 1]$.

3.3. Example: [2]

Let A be a fuzzy subset of $\mathbb{R}^3$ such that $A(a, b, c) = 1$ for all $(a, b, c) \in \mathbb{R}^3$ and B be a fuzzy subset of $\mathbb{R}^2$ such that $B(a,b) = 1/2$ for all $(a,b) \in \mathbb{R}^2$. $K : A \to B$ such that $K(a,b,c) = (a,b)$, for all $(a,b,c) \in \mathbb{R}^3$. K is a fuzzy linear transformation on a fuzzy subspace A.

3.1. Proposition: [2]

Let A be a fuzzy subspace of a vector spaces U over F and B be a fuzzy subspace of a vector space V over F.Let $K : A \to B$ be a fuzzy linear transformation if and only if $T : U \to V$ is a linear transformation on vector space.

Proof:

$\Rightarrow$ Since $K : A \to B$ is fuzzy linear transformation, that mean:

$K(\lambda x_{t_1}, \alpha y_{t_2}) \geq \min\{K(x_{t_1}), K(y_{t_2})\}$,for all $x_{t_1}, y_{t_2} \in A$ and $\lambda, \alpha \in F$, $t_1, t_2 \in [0,1]$.

To prove $T : U \to V$ is a linear transformation on vector space, (i.e.) $T(\lambda x + \alpha y) = \lambda T(x) + \alpha T(y)$, for all $x, y \in U$ and $\lambda, \alpha \in F$.

Since $x_{t_1}, y_{t_2} \in A$, $t_1, t_2 \in [0,1]$, then there exists $x, y \in U$ such that $K(x_{t_1}) = T(x)$, $K(y_{t_2}) = T(y)$ implies that $x = T^{-1}(K(x_{t_1}))$ and $y = T^{-1}(K(y_{t_2}))$.

$$\begin{aligned}
T(\lambda x + \alpha y) &= T(\lambda x) + T(\alpha y) \\
&= T(\lambda T^{-1}(K(x_{t_1}))) + T(\alpha T^{-1}(K(y_{t_2}))) \\
&= \lambda T(T^{-1}(K(x_{t_1}))) + \alpha T(T^{-1}(K(y_{t_2}))) \\
&= \lambda K(x_{t_1}) + \alpha K(y_{t_2}) \\
&= \lambda T(x) + \alpha T(y)
\end{aligned}$$

,for all $x, y \in U$.

Then $T(\lambda x + \alpha y) = \lambda T(x) + \alpha T(y)$, for all $x, y \in U$ and $\lambda, \alpha \in F$.

Hence $T : U \to V$ is a linear transformation on vector space.

$\Leftarrow$ Since $T : U \to V$ is a linear transformation on vector space, that mean:

$T(\lambda x + \alpha y) = \lambda T(x) + \alpha T(y)$, for all $x, y \in U$.

To prove $K : A \to B$ is fuzzy linear transformation, (i.e.) $K(\lambda x_{t_1}, \alpha y_{t_2}) \geq \min\{K(x_{t_1}), K(y_{t_2})\}$,for all $x_{t_1}, y_{t_2} \in A$ and $\lambda, \alpha \in F$, $t_1, t_2 \in [0, 1]$.

Since $\lambda, \alpha \in F$ and $x_{t_1}, y_{t_2} \in A$, $t_1, t_2 \in [0, 1]$ such that $K(x_{t_1}) = T(x)$, $K(y_{t_2}) = T(y)$ implies that $x = T^{-1}(K(x_{t_1}))$ and $y = T^{-1}(K(y_{t_2}))$.

$$\begin{aligned}
K(\lambda x_{t_1}, \alpha y_{t_2}) &= K(\lambda x_{t_1}) + K(\alpha y_{t_2}) \\
&= K(\lambda K^{-1}(T(x))) + K(\alpha K^{-1}(T(y))) \\
&= \lambda K(K^{-1}(T(x))) + \alpha K(K^{-1}(T(y))) \\
&= \lambda T(x) + \alpha T(y) \\
&= \lambda K(x_{t_1}) + \alpha K(y_{t_2}), \text{ for all } x, y \in U,\ \lambda, \alpha \in F. \\
&\geq K(x_{t_1}) + K(y_{t_2}), \text{ for all } x, y \in U,\ \lambda, \alpha \in F.
\end{aligned}$$

$K(\lambda x_{t_1}, \alpha y_{t_2}) \geq K(x_{t_1})$ and $K(\lambda x_{t_1}, \alpha y_{t_2}) \geq K(y_{t_2})$.

$K(\lambda x_{t_1}, \alpha y_{t_2}) \geq \min\{K(x_{t_1}), K(y_{t_2})\}$,for all $x_{t_1}, y_{t_2} \in A$ and $\lambda, \alpha \in F$, $t_1, t_2 \in [0, 1]$.

Hence $K : A \to B$ is fuzzy linear transformation.

3.3. Theorem: [2]

Let A, B, C be fuzzy subspaces of vector spaces U, V, W over F respectively and let $K : A \to B$, $G : B \to C$ be fuzzy linear transformations. Then $G \circ K : A \to C$ be a fuzzy linear transformation.

Proof:

Since K and G are fuzzy linear transformations, then:

$K(\lambda x_{t_1} + \alpha y_{t_2}) = \sup\{\inf\{\lambda, K(x_{t_1}), \alpha, K(y_{t_2})\}$, for all $x_{t_1}, y_{t_2} \in A$ and $\lambda, \alpha \in F$, $t_1, t_2 \in [0, 1]$.

$G(\lambda z_{t_3} + \alpha u_{t_4}) = \sup\{\inf\{\lambda, G(z_{t_3}), \alpha, G(u_{t_4})\}$, for all $z_{t_3}, u_{t_4} \in B$ and $\lambda, \alpha \in F$, $t_3, t_4 \in [0, 1]$.

and $K(x_{t_1}) = z_{t_3}, K(y_{t_2}) = u_{t_4}, G(z_{t_3}) = a_{t_5}, G(u_{t_4}) = b_{t_6}$, $a_{t_5}, b_{t_6} \in A$, $t_5, t_6 \in [0, 1]$.

To prove $G \circ K : A \to C$ be a fuzzy linear transformation, let $a_{t_5}, b_{t_6} \in A$ (that mean $a, b \in U$ and $t_5, t_6 \in [0, 1]$), for all $\lambda, \alpha \in F$, then:

$$\begin{aligned} G \circ K(\lambda a_{t_5} + \alpha b_{t_6}) &= G(K(\lambda a_{t_5} + \alpha b_{t_6})) \\ &\geq G(\min\{K(a_{t_5}), K(b_{t_6})\}) \\ &= G(\min\{z_{t_3}, u_{t_4}\}) \\ &= \min\{G(z_{t_3}), G(u_{t_4})\} \\ &= \min\{G(K(x_{t_1})), G(K(y_{t_2}))\} \\ &= \min\{G \circ K(a_{t_5}), G \circ K(b_{t_6}) \end{aligned}$$

Then $G \circ K : A \rightarrow C$ be a fuzzy linear transformation.

3.2. Proposition: There is a forgetful functor from fuzzy normed spaces and fuzzy linear transformations category to vector spaces and linear transformations category.

Proof:

Let the vector spaces and linear transformations category indicated by VS. Then associated with the objects set of VS: $ob(VS)$ and the linear transformations set of VS: for $\forall U_1, U_2 \in ob(VS)$, $f_1 = [U_1, U_2]_{VS} \in MorVS$; this category provides cat_1 and cat_2 .

Similarly fuzzy normed spaces and fuzzy linear transformations category FN_L provides cat_1 and cat_2, associated with $ob(FN_L)$ and $MorFN_L$.

Let (U_1, N_1) such that $N_1 : U_1 \times R \rightarrow [0,1]$ and likewise (U_2, N_2), $(U_3, N_3) \in ob(FN_L)$, and $T_1, T_2 \in MorFN_L$. Then we can describe a forgetful functor F

$$\boxed{(U_1, N_1) \overset{T_1}{\rightarrow} (U_2, N_2) \overset{T_2}{\rightarrow} (U_3, N_3)}$$
$$\downarrow F$$
$$\boxed{U_1 \overset{f_1}{\rightarrow} U_2 \overset{f_2}{\rightarrow} U_3}$$

This functor causes to forget the fuzzy norm of $ob(FN_L)$, and the fuzzy feature of $MorFN_L$ from proposition 3.1..

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